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ON THE KINETIC THEORY OF STEADY GAS FLOWS

by

Lawrence Sirovich

June 28, 1960

AEC Research and Development Report

NEW YORK UNIVERSITY

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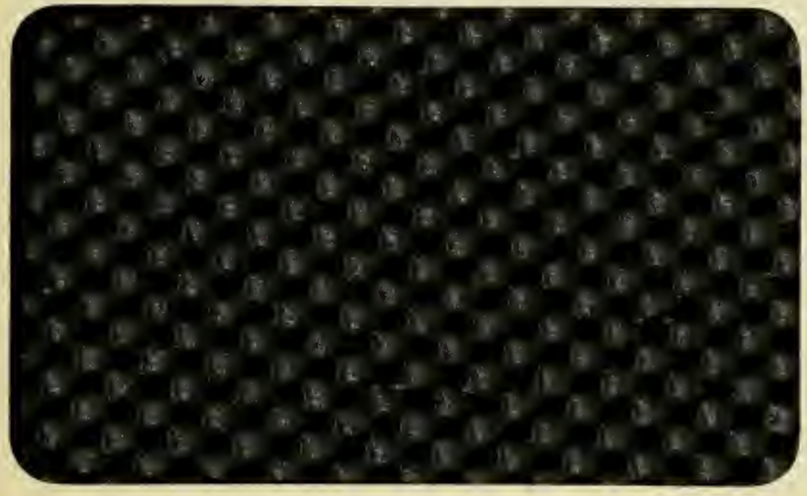

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Magneto-Fluid Dynamics Division
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MATHEMATICS

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ABSTRACT

An approximate Boltzmann equation, known as the single relaxation model [4], is studied here. This equation is linearized and the fundamental solution is considered. Following H. Grad, [6], the solution, asymptotic in small values of the ratio of mean-free-path to distance from the origin, is sought. It can be shown that the fundamental solution itself gives the asymptotic description of the flow field past an object. This solution gives the asymptotic description when the distance from the origin is much greater than either the mean-free-path or the body size. This is true independently of the Knudsen number.

It is shown that within the linearized framework used, the Navier-Stokes equations give a consistent asymptotic description of the correct kinetic theory flow field. On the other hand, the inviscid Euler equations fail in the neighborhood of the Mach lines. The asymptotic fundamental solution found gives the fine structure of the wake uniformly in the Mach number. In addition, the solution gives a detailed description of the neighborhood of the Mach line, uniformly in Mach number, as the latter varies from supersonic to high subsonic values. The uniform solution is given in closed form in terms of an Airy function.

In the limit of zero mean-free-path, the fundamental solution of the inviscid equations is recovered. Convergence to the inviscid solutions is not uniform.

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PREFACE

The problem discussed in this report was suggested by Harold Grad. It is with warmth and pleasure that the author acknowledges his gratitude to him, not only for the suggestion of the problem, but also for the many stimulating and inspiring discussions. In this connection the author is also grateful to Dr. Harold Weitzner for his helpful suggestions and discussions, especially in connection with the theory of generalized functions, and to Dr. Clifford Gardner who willingly gave his help.

ON THE KINETIC THEORY OF STEADY GAS FLOWS

1. Introduction.

In recent years there has been an increasing interest in the problems of kinetic theory. Much of this renewed interest can be traced to the work of Wang Chang and Uhlenbeck [7], [14], and later Mott-Smith [20]. Using approximate Boltzmann equations and exploiting the tools of linearized theory, they were able to deal with some special but fundamental problems, such as sound propagation, Couette flow, and heat flow. More recently attention has been paid to kinetic models. By a kinetic model we mean an equation which is intended to mimic the Boltzmann equation in as many ways as possible, and yet remain analytically simple. A model which seems to have first appeared in the literature in a work of Welander [1] and simultaneously in a paper by Bhatnager, Krook and Gross [2], has proven successful in a number of problems, see [2], [4], [5]. The modeling lies in the replacement of the Boltzmann collision operator by a term proportional to the deviation of the distribution from the locally Maxwellian distribution function

$$\frac{\partial f}{\partial t} + \xi \cdot \frac{\partial f}{\partial x} = \nu(f_o - f).$$

The constant of proportionality, ν, is the collision frequency; the remainder of the notation is that commonly used and is further explained in Section 3. This equation exhibits the more important features of the full Boltzmann equation. Briefly, moments of the collision term vanish when weighted by the collisional invariants, so that the conservation equations follow, furthermore the H theorem is valid; for details see [8]. It is also clear from the model that a relaxation theory (the space homogeneous case) predicts that all moments of the distribution function, other than density, velocity, temperature, decay with the same period. For this reason, the equation is referred to as the single relaxation model [4]. Quite recently H. Grad [6] considered the fundamental solution of the single relaxation model. He developed a technique for gaining an asymptotic form of the fundamental solution. This is fully discussed in Section 3.

Before going on further, we digress a moment to discuss what is meant by the linearized Boltzmann equation. As is shown in [4], our equation can be derived from the Boltzmann equation in which squares of the perturbed distribution function away from the local Maxwellian are neglected. This equation is often referred to as the "linearized" Boltzmann equation. Yet, it is far from being linearized, since moments of the distribution function occur in the local Maxwellian (a quadratic non-linearity is replaced by an exponential non-linearity). We shall refer to this linearization as the weak linearization, and reserve the appellation of strong linearization for a kinetic equation which is linearized away from an absolute Maxwellian. Aside from the reference cited above, another recipe for obtaining the single relaxation model (from the exact non-linear Boltzmann equation in this case) can be found in [3]. No consideration of these derivations will be found in this work; rather if one wishes, the merits of the model may be evaluated in light of the subsequent theory it predicts.

In Section 3, a procedure which resembles the Chapman-Enskog method [16], is applied to the steady strongly linearized form of our equation (also see the discussion in the next section). The hierarchy of linearized hydrodynamical equations is found to be: the linearized Euler equations (Sec. 3), the Oseen equations (see Section 5), a system formally equivalent to the linearized Burnett equations, and higher moment systems (see Section 6). As is indicated in Section 5, the computed viscosity coefficient is "correct"*, however, the heat conductivity is off by a factor of 3/2. It may seem rather remarkable that the Navier-Stokes system should result after so much has been done to destroy the character of the collision operator. The implication is that the Naview-Stokes system is relatively insensitive to the collision process, and depends on only those gross properties which have been included in the single relaxation model.

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* The viscosity (or the conductivity, but not both) can be made anything one wishes by the appropriate choice of ν, the "collision frequency". A consistent approach, as is given in [4] indicates the choice of ν. This leads to a viscosity coefficient, which Chapman-Cowling [16], call the first approximation.

In the above introduction, we have anticipated some of the results which will be found later. In the section which follows, in addition to interpretation, we give a verbal picture of the results of this report. A detailed study of the structure of certain simple flows is given as well as application to more complicated flows. No attempt at proving statements is made; however, reference to later discussion is given parenthetically.

2. Survey.

As mentioned in the preceeding section, a kinetic model will be used in place of the full Boltzmann equation. However, the preliminary remarks made in this section apply also to the full Boltzmann equation. For this reason we start the discussion in generality and only later particularize.

In order to sharpen the succeeding discussion, let us focus attention on a particular problem. We consider the flow past a particle source located at the origin. The source can be thought of as a point in space, which spews out and absorbs particles. To further fix the picture, we take the particle source such that it conserves the number of particles, i.e., it is not a mass source. Further, we endow it with a lift, drag, and heat generation. Higher moments such as stress, heat flow, etc., could also be associated with the particle source, but these will not enter the discussion which follows. If the equation governing the flow is taken to be linear, the proper superposition of these sources can be found (in principle) to give the flow past some arbitrarily shaped body. For our strongly linearized equation, this is the problem of finding the fundamental solution (see Section 3). For the present, it is not necessary to be this restrictive.

The Boltzmann equation for a simple gas contains only one pertinent length scale, the mean free path, L. We gain tractability if instead of seeking a general solution, we look for a solution which is asymptotic in either small L/R or small R/L. R is the distance from the origin. A solution corresponding to the latter limit describes the flow field within a mean free path of the origin, where we have essentially "free molecular" flow. An application of this solution in the linear case would be the superposition of sources along a wall to give a description of the slip flow region adjacent to the wall. Incidentally, in this regime the collision term of the full Boltzmann equation, which contains the non-linearity, is a higher order term, so that the superposition of sources would presumably still give an accurate picture.

It is the former limit, that of small L/R, with which we will be concerned in the main body of this work. A solution corresponding to this limit depicts the flow field many mean free paths from the origin. Since the gas many mean free paths from boundaries has had the opportunity to relax, the distribution function will be nearly Maxwellian. We then would expect the deviation from the Maxwellian to be given in terms of the density, temperature, velocity, and their derivatives. That the description will be hydrodynamical (i.e., in terms of density, temperature, and velocity) may be seen in another way. With proper normalization of the steady Boltzmann equation, the Chapman-Enskog expansion is an expansion in the ratio L/R, supposedly valid for small L/R. This method, as is well known, leads to a hierarchy of fluid dynamical equations. Although the language used is fluid mechanical, it is important to note that the above remarks refer to the entire Knudsen number range. When sufficiently far away from a body, a hydrodynamical description may be given, independent of the rarity of the gas.

At this point we leave the general case to place restrictions on the governing equation and the cases considered. We choose the steady form of the strongly linearized single relaxation model (see equation (3.10)). The constant flow at negative infinity is governed by the absolute Maxwellian about which linearization takes place. Linearization places a rather strong restriction on a solution. We know from experience that linearization is incorrect within shock waves and the Prandtl boundary layer. To avoid repetitious reference to the linearization requirement, we adopt the point of view that we are dealing with a gas which behaves in accordance with the linearized equation. Then at the end of the discussion we can make the all-encompassing declaration that the solution is not valid in regions of the flow where quadratic terms are not negligible with respect to linear terms.

Under the restrictions of the preceeding paragraph, the problem set forth is equivalent to finding the fundamental solution (see Section 3). H. Grad showed that to the first approximation (in L/R), this reduces to finding the fundamental solution of the linearized Euler equations. In anticipation of what follows, it will serve our purposes if we pause to consider what is meant by first approximation, second approximation, etc. The first approximation comes merely from the grouping of terms of O(L/R) and ignoring the higher orders (see Section 3); the second approximation consists of grouping terms of O(L/R) and $O(L^2/R^2)$ and ignoring the rest (see Section 5), and so on. This procedure results in a hierarchy of equations in density, temperature and velocity. As mentioned in the introduction, this results in the linearized Euler equations, the Oseen equations, and then higher moment systems.

Although grouping terms in this manner seems computationally attractive, there is no compelling reason to do so. Rather than a rule, it should act as a guide. We now elucidate this comment. Solving for the fundamental solutions of the linearized Euler equations, we get the first approximation to the fundamental solution of the strongly linearized single relaxation model. This leads, in the case of supersonic flow, to singularities along the Mach cone, see [6] (also see Sec. 4). Further, H. Grad shows in reference [6], that there is another singularity which is along the flow direction behind the source. When properly interpreted, this can be identified with the wake in the inviscid limit (Sec. 4). The latter singularity exists through subsonic velocities (Sec. 4). In the neighborhood of these singularities higher order terms in (L/R) are of the same effective magnitude as those terms in the first approximation (Sec. 6). For this reason the grouping according to O(L/R) is not valid throughout the flow field. One finds that the terms which must be carried to correctly describe the "singular regions" (they are no longer singular with the additional terms) belong to the second approximation, i.e., they come from the grouping which yields the Oseen equations (Sec. 5). Furthermore, the terms added by higher approximations are indeed of higher order, and in particular this is true for the terms which come from the "Burnett" equations (Sec. 6). This, within the linearized kinetic theory framework laid down here, substantiates the use of the Navier-Stokes equations in fluid mechanics. Actually the association with hydrodynamical equations need never have been made, in the sense that it has no bearing on the analysis. One can formally carry out the analysis even if in complete ignorance of fluid mechanics (at least in principle). It should again be noted, that although we have been using the language of fluid mechanics (implying small Knudsen number), the results and analysis apply to the entire spectrum of Knudsen number. This is so for the simple reason that the fundamental solution is applicable to bodies of any size and hence independent of the usually defined Knudsen number. There is however, if we choose to define it, a Knudsen nymber based on mean free path compared to distance from the source, and we are looking for solutions for small values of this ratio. Once we have the fundamental solution for small Knudsen number based on distance, we can apply it to problems for all Knudsen numbers based on mean free path compared to body size.

Before discussing the actual nature of the solutions, we consider the application of the fundamental solution. In general, in order to solve for flow past a specific body, we cannot avoid the travail of summing sources and solving the resulting integral equations. If we are successful with the mathematics, a detailed picture of flow is given outside of a region of several mean free paths surrounding the body. If we choose to be less demanding in the amount of information that we want, we find a cache of information directly from the fundamental solution itself. It is shown in Section 3 that for given values of drag, lift and heat addition, the fundamental solution, with these values inserted, gives the asymptotic flow field. This solution is asymptotic in the two Knudsen ratios mentioned in the preceeding paragraph. For objects whose dimensions are of the order of the mean free path or smaller we would presume that the fundamental solution begins to describe the flow many mean free paths from the body. For objects whose dimensions are larger than the mean free path (i.e., for those flows which embrace fluid mechanics) we must place ourselves many body lengths from the object before the fundamental solution begins to describe the flow field. In either case if the problem is solved by summing sources, the two solutions approach each other asymptotically as we allow R to increase (Sec. 3). We should bear in mind that in the former case, that of using the fundamental solution directly, we must be content with being ignorant of the lift, drag and heat transfer, whereas in the latter case the lift, drag and heat transfer is computed from the calculated flow field.

We now turn to the approximate fundamental solutions which are determined in Sections 7 and 8. The analytical results which are mentioned refer to the two-dimensional case. This is the case dealt with in this report. Let us note that since the terms used in the computation come from the Oseen grouping, we are also discussing the fundamental solution of the Oseen equations. The Mach layer solutions of Section 8 are uniformly valid with respect to Mach number, as we pass from supersonic to high subsonic velocities. Section 7 contains the wake calculation which is uniformly valid for all Mach numbers. Aside from the wake no higher approximation is necessary for flows up to high subsonic velocities. For these cases the inviscid solutions themselves give a consistent picture outside the wake. In discussing the results of Sections 7 and 8, we deal separately with transonic and supersonic flow. A fluid dynamical approach to the supersonic Oseen equations has been considered by Lagerstrom, Cole, Trilling [18], also Cole and Wu [19]. Although different methods were used, there is agreement in the results.

For supersonic flow, the Mach lines, and the wake line of inviscid theory act as a skeleton for the higher order flow field. The inclusion of higher order terms fuzzes out these lines. We find that the diffusive region surrounding the Mach lines is centered on the Mach line and grows with the distance along the Mach line, h, as $(Lh)^{1/2}$ (Sec. 6 and 8). Furthermore, the fine structure of this layer is governed by the similarity variable (t^2/hL), where t is the coordinate normal to the Mach line (Sec. 6 and 8). A diffusion operator is responsible for the Mach layer, and, as we would expect, a diffusion operator also governs the wake (Sec. 6). The wake spreads as $(Lx)^{1/2}$, where x is the downstream variable, and the fine structure of the wake is governed by the similarity variable (y^2/Lx), y being the transverse variable (Sec. 7). Allowing the mean free path to vanish, we recover the inviscid solutions (Sec. 4, 7, 8). In doing this we go from a parabolic to a hyperbolic partial differential equation (Sec. 6), so that the singular lines exist as unveiled characteristics. A most important feature of the limiting process is that the convergence is not uniform, it being slower more distant from the origin.

To gain insight into this non-uniform convergence, we look at it in another but equivalent way. In looking at the supersonic solution, (Sec. 8), and also from remarks above, we see that h and L are interchangeable. That is, allowing h to vanish is equivalent to a vanishing L. But as we mentioned before (also Sec. 8) a vanishing L gives us the inviscid solutions, and also therefore, so does a vanishing h. So that for a small but fixed L, as we allow h to approach zero, the solution more closely resembles the inviscid solutions. This fortifies the idea that inviscid flow occurs just outside the boundary layer. However, no matter how small $L > 0$ is chosen, we can go far enough away from the origin (and hence a body), such that the Mach lines are diffused, and the inviscid solution does not give an adequate description of the flow there. We are now in a position to give an over-all picture of supersonic flow according to the linearized framework we have adopted. Starting from the body we first have a boundary layer region[*] (as we have mentioned earlier, the slip flow region cannot be described by our analysis) which changes to the wake as we go downstream. Outside of the boundary layer, the flow becomes inviscid and remains so for distances which are $O(\mathcal{L}^2/L)$. This is the distance at which the layer is approximately double the body size. Sufficiently far away, the region in which disturbances propagate, i.e., the Mach region, and the wake (which grow as $(Lh)^{1/2}$, $(Lx)^{1/2}$ respectively) widen enough so that a viscous description is again needed. With the exception of the Mach line and wake line, any ray from the origin ultimately passes into and stays in a region of inviscid flow. On the other hand, any parallel to Mach lines or the wake line ultimately passes into and stays in a viscous region.

A similar set of circumstances also holds for the transonic case; however the results are somewhat more striking. Consideration of the wake is independent of Mach number and the discussion of it is not repeated. For simplicity only sonic flow will be described. Again we find a diffusive layer centered on the Mach line, which in this case is the y-axis. The layer thickness grows as $(y^2L)^{1/3}$ (Sec. 8) which is more rapid than in the supersonic case. Detailed information of the structure of the layer is now described by the similarity variable (x^3/y^2L) (Sec. 8). The differential operator which governs the layer is third order parabolic, and not of a well-known form. Perhaps the most interesting question that can be asked of the sonic solution is its limit for a vanishing mean free path. As we show in Section 8, this is the much maligned solution of the inviscid sonic equation (Sec. 4). The latter violates the linearization restriction, the velocity perburbation grows linearly in y, and predicts an infinite drag because of it (Sec. 4). The clue to the situation lies in the non-uniform convergence of the solution as L is allowed to become small. No matter how small L is made, sufficiently far away the viscous solution takes over and decays as $1/y^{1/3}$. The drag then is bounded, see Section 8.

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[*] Even though we have a linearized theory, a boundary layer is formed, see for instance [19].

In the last two sections we sketched the flows for supersonic and just sonic speeds. Sections 7 and 8 give the uniform solution from which the sketches were taken. It is exhibited in closed form in terms of an Airy function. The analysis of Section 6 shows that the solution is also applicable to high subsonic velocities.

3. Governing Equations.

For the sake of preliminary discussion, we shall discuss the steady flow about an arbitrary body by means of kinetic theory. As a governing equation, we choose the single relaxation time kinetic model ([1], [2]),

$$(3.1) \qquad \xi \cdot \nabla f = \nu(f_o - f),$$

which has been studied by Gross and his co-workers ([3], [4], [5]). The distribution function, f, is normalized to be a mass density. The collision frequency, ν, is proportional to the mass density. f_o denotes the locally Maxwellian distribution function described below, and ξ is the molecular velocity. We will not consider the approximations needed to arrive at (3.1), since this is discussed in the literature ([3], [4]). Besides, before very long we shall see equation (3.1) in light of the description it produces, and may evaluate it from that standpoint.

H. Grad [6] studied the fundamental solution of (3.1), i.e., the solution of

$$(3.2) \qquad \xi \cdot \nabla f = \nu(f_o - f) + \sigma(\xi)\delta(x).$$

The source term $\sigma(\xi)$, will be discussed presently. Grad developed a technique for gaining an asymptotic form of the fundamental solution, and the present problem, suggested by him, is in a sense, to be considered the next step in the asymptotic development.

The single relaxation model, simple as it seems, is still a non-linear integro-differential equation. This is so, by virtue of the local Maxwellian, f_o, which contains the five molecular velocity moments of the distribution function, f, giving the local density, temperature, and velocity, $\rho(x)$, $T(x)$ and $u(x)$.

$$(3.3) \qquad f_o = \frac{\rho(x)}{[2\pi RT(x)]^{3/2}} e^{[\xi - u(x)]^2/2RT(x)}$$

where

$$(3.4) \qquad \rho \begin{bmatrix} 1 \\ u \\ RT \end{bmatrix} = \int f \begin{bmatrix} 1 \\ \xi \\ \frac{(\xi-u)^2}{3} \end{bmatrix} \bar{d}\xi$$

As is shown in reference [3], the single relaxation model can result in a certain way from the weakly linearized Boltzmann equation; however, it still leads to a non-linear hydrodynamic theory, as can be seen by taking velocity moments of (3.1). Following Grad [6] we write

$$(3.5) \qquad \begin{cases} f = f^o + g \\ \rho = \rho_o + \rho \\ T = T_o + T \\ p = p_o + p \\ u = u_o + u \end{cases}$$

where

$$(3.6) \qquad f^{o} = \frac{\rho_o}{[2\pi RT_o]} e^{-(\xi - u_o)^2/2RT_o}$$

is the unperturbed constant flow at infinity. Using the same letter for a quantity and its perturbation should not lead to confusion since the letter will henceforth denote just the perturbation. Denoting the relative molecular velocity by

$$(3.7) \qquad c = \xi - u_o$$

we have

$$(3.8) \qquad \begin{cases} \rho = \int g \, d\xi \\ \rho_o u + \rho u_o = \int \xi g \, d\xi \\ \rho_o u = \int c g \, d\xi \\ p = \frac{1}{3} \int c^2 g \, d\xi \\ \rho_o RT = \int (\frac{1}{3} c^2 - RT_o) g \, d\xi \\ p = \rho + T \end{cases}$$

Linearizing the local Maxwellian about the absolute Maxwellian

$$(3.9) \qquad f_o \sim f^{(0)} \left\{ 1 + \frac{\rho}{\rho_o} + \left(\frac{c^2}{2RT_o} - \frac{3}{2}\right) \frac{T}{T_o} + \frac{c \cdot u}{RT_o} \right\},$$

and introducing the above into equation (3.1), we get

$$(3.10) \qquad \xi \cdot \nabla g + \nu g = \nu f^{(0)} \left\{ \frac{\rho}{\rho_o} + \left(\frac{c^2}{2RT_o} - \frac{3}{2}\right) \frac{T}{T_o} + \frac{c \cdot u}{RT_o} \right\}.$$

Next introducing dimensionless quantities

$$(3.11)\quad \begin{cases} \tilde{\rho} = \dfrac{\rho}{\rho_o} \;, \quad \tilde{T} = \dfrac{T}{T_o} \;, \quad \tilde{p} = \dfrac{p}{p_o} \\ \tilde{u} = \dfrac{u}{\sqrt{RT_o}} \;, \quad \tilde{u}_o = \dfrac{u_o}{\sqrt{RT}} \;, \quad \tilde{\xi} = \dfrac{\xi}{\sqrt{RT_o}} \\ \tilde{c} = \dfrac{c}{\sqrt{RT_o}} \\ \tilde{g} = (RT_o)^{3/2}\, g/\rho_o \\ \omega = \dfrac{1}{(2\pi)^{3/2}}\, e^{-\frac{1}{2}c^2} \end{cases}$$

and the mean free path, L, by

$$(3.11')\quad \frac{1}{L} = \frac{\nu}{\sqrt{RT_o}}$$

we have from (3.8)

$$(3.12)\quad \begin{cases} \tilde{\rho} = \int \tilde{g}\, d\xi \\ \tilde{u} = \int \tilde{c}\tilde{g}\, \overline{d}\tilde{\xi} \\ \tilde{p} = \int \frac{1}{3}\, \tilde{c}^2\tilde{g}\, \overline{d}\tilde{\xi} \\ \tilde{T} = \int (\frac{1}{3}\, \tilde{c}^2 - 1)\tilde{g}\, \overline{d}\tilde{\xi} \end{cases}$$

and (3.1) now becomes

$$(3.13)\quad \xi \cdot \frac{\partial g}{\partial x} + \frac{1}{L}\, g = \frac{\omega}{L} \left\{ \tilde{\rho} + (\frac{1}{2}\, \tilde{c}^2 - \frac{3}{2})T + \tilde{c}\cdot\tilde{u} \right\} .$$

As a result of the linearization, ν and hence L, are now constants. This is the equation that H. Grad ultimately studied in reference [6].

For the moment we make the spatial variable, x, dimensionless by comparing it with R, which is representative of the distance from an observer point to the origin.

$$(3.14) \qquad x/R = x' .$$

With this stretching, equation (3.13) becomes

$$(3.15) \qquad \epsilon\tilde{\xi} \cdot \frac{\partial}{\partial x'} g + g = \omega \left\{ \tilde{\rho} + \left(\frac{1}{2}\tilde{c}^2 - \frac{3}{2}\right)\tilde{T} + \tilde{c}\cdot\tilde{u} \right\}$$

where

$$\epsilon = L/R .$$

Taking the Fourier transform of equation (3.15) and denoting the transformed variable by the untransformed symbol, for instance

$$(3.16) \qquad \tilde{\rho}(k) = \int \tilde{\rho}(x)\, e^{-ik\cdot x}\, \overline{dx}$$

we get

$$(3.17) \qquad \tilde{g}(k,\tilde{\xi}) = \frac{\omega}{1 + \epsilon ik\cdot\xi} \left\{ \tilde{\rho} + \left(\frac{1}{2}\tilde{c}^2 - \frac{3}{2}\right)\tilde{T} + \tilde{c}\cdot\tilde{u} \right\} -$$

$$- \frac{\epsilon}{1 + \epsilon ik\cdot\xi} \int_{\Sigma} g e^{-ikx'}\, \xi\cdot ds \quad .$$

The integral is over the surface of the body, Σ, the normal is directed into the body.* It arises by use of the divergence theorem, from the parts integration of the Fourier transform of the convective term of the differential term. The argument of the exponent, x', is $O(\mathcal{L}/R)$ where $\mathcal{L}$ is a length representative of the length of the body. Expanding the exponential, we get an asymptotic representation of the integral in the small parameter $(\mathcal{L}/R)$

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* The surface integral can be taken at the body only if the object is slender, and the boundary conditions allow linearization. If this is not the case, the surface integral is to be taken at some surface surrounding the body at which the linearization holds.

$$(3.18) \qquad \int_{\Sigma} g e^{-ikx'} \xi \cdot ds \sim \int_{\Sigma} g\xi \cdot ds - \int_{\Sigma} ik \cdot x' \, g\xi \cdot ds + \ldots$$

The leading term of equation (3.18) is $O(\mathcal{L}^2/R^2)$, by virtue of being a surface integral. At this point we have two small dimensionless ratios, L/R and $\mathcal{L}/R$. There is no reason to compare them in the analysis. The first ratio $\epsilon = L/R$, as we will see shortly after expanding in powers of it, acts as a guide in choosing approximate equations; whereas the second ratio $\mathcal{L}/R$ as we see from equation (3.18) splits the boundary conditions into orders of $\mathcal{L}/R$. In what follows we shall carry only the first term of the expansion (3.18). This then yields a solution which is asymptotic in the ratio $\mathcal{L}/R$. By virtue of the forthcoming expansion in ϵ, this solution is asymptotic in the two ratios $\mathcal{L}/R$ and L/R. The expansion in $\epsilon = L/R$ demands that the solution be used only many mean free paths away.

Returning to equation (3.17), retaining only the first term of the integral, (3.18), and removing the stretching (3.14), we get

$$(3.19) \qquad \tilde{g}(k,\xi) = \frac{\omega}{1+iLk\cdot\tilde{\xi}} \left\{ \tilde{\rho} + \left(\frac{1}{2}\,\tilde{c}^2 - \frac{3}{2}\right)\tilde{T} + \tilde{c}\cdot\tilde{u} \right\} + \frac{L\sigma(\xi)}{1+iLk\cdot\tilde{\xi}}$$

with

$$(3.20) \qquad \tilde{\sigma}(\xi) = - \int_{\Sigma} \tilde{g}(x,\tilde{\xi})\tilde{\xi}\cdot ds.$$

Henceforth, when expanding with respect to the mean free path L, we shall mean that L is small in comparison to the radial distance of an observer point. From (3.20) we see that $\tilde{\sigma}(\xi)$ is the mass flow out of the body per volume of velocity space (this is not dimensionally true, due to the dimensionalization (3.11)). In addition we see that (3.19) is precisely the equation we would get by our procedure if we had considered Grad's equation, (3.2), to begin with. We therefore, see that solution of (3.19) applies to two different cases. First it gives the asymptotic solution for flow past a finite object subject to the preceding remarks. Second, it gives a fundamental solution for the strongly linearized single relaxation kinetic model. This distinction is not just a matter of interpretation. Since we have linearized our equation, the solution, when considered as applied to the asymptotic gas flow past an object, must agree with the linearization. Hence it applies only to those regions where products of the hydrodynamical quantities are small compared to the quantities themselves. This restriction is not to be invoked on the solution when it is viewed as a fundamental solution. The fundamental solution is used to construct solutions of boundary value problems, and it is only the latter solutions which must comply with the linearization requirement. Furthermore, the latter case places no restrictions on the dimensions of the object. In fact we can use the fundamental solution to construct solutions of boundary value problems through all ranges of Knudsen number.

On multiplying equation (3.19) by 1, $\tilde{c}$, $(\frac{1}{3}\tilde{c}^2 - 1)$ and integrating in velocity space we get an inhomogeneous linear system of equations for $\tilde{\rho}$, $\tilde{T}$, $\tilde{u}$, which Grad [6] first derived. The equations are

$$(3.21)\quad \begin{pmatrix} 1 - \int \Omega d\tilde{\xi} & -\int \frac{1}{2}(\tilde{c}^2-3)\Omega d\tilde{\xi} & -\int \tilde{c}_s \Omega d\xi \\ -\int \frac{1}{3}(\tilde{c}^2-3)\Omega d\tilde{\xi} & 1-\int \frac{1}{6}(\tilde{c}^2-3)^2 \Omega d\tilde{\xi} & -\int \frac{1}{3}\tilde{c}_s(\tilde{c}^2-3)\Omega d\tilde{\xi} \\ -\int c_r \Omega d\tilde{\xi} & -\int \frac{1}{2}\tilde{c}_r(c^2-3)\Omega d\tilde{\xi} & 1-\int \tilde{c}_r c_s \Omega d\tilde{\xi} \end{pmatrix} \begin{pmatrix} \tilde{\rho} \\ \tilde{T} \\ \tilde{u}_s \end{pmatrix} =$$

$$= L \begin{pmatrix} \tilde{\sigma}_1 \\ \tilde{\sigma}_2 \\ \tilde{\sigma}_3 \end{pmatrix}$$

with

$$(3.22)\quad \begin{cases} \Omega = \dfrac{\omega}{1+iLk\cdot\tilde{\xi}} \\ \tilde{\sigma}_1 = \displaystyle\int \frac{\tilde{\sigma} d\xi}{1+iLk\cdot\tilde{\xi}} \\ \tilde{\sigma}_2 = \displaystyle\int \frac{(1/3\ \tilde{c}^2-1)\sigma d\tilde{\xi}}{1+iLk\cdot\tilde{\xi}} \\ \tilde{\sigma}_3 = \displaystyle\int \frac{\tilde{c}\ \tilde{\sigma}\ d\tilde{\xi}}{1+iLk\cdot\tilde{\xi}} \end{cases}$$

The matrix equation (3.21) is a 5×5 system, repeated subscripts representing summation, and unrepeated subscripts ranging through the number of dimensions of configuration space.

In order to perform the indicated integrations, we expand

$$(3.23) \qquad \Omega \sim \omega \left\{ 1 - iLk\cdot\tilde{\xi} + L^2(k\cdot\tilde{\xi})^2 + \dots \right.$$

The resulting integrated series gives the asymptotic representation of the integrals occuring in (3.21). Retaining only the first term, i.e., dropping O(L), equation (3.21) identically vanishes. Carrying terms of O(L), i.e., taking

$$(3.24) \qquad \Omega \sim \omega\,(1 - iLk\cdot\tilde{\xi})$$

we get

$$(3.25) \qquad \begin{pmatrix} i\tilde{u}_o\cdot k & 0 & ik_s \\ 0 & i\tilde{u}_o\cdot k & \frac{2}{3}\,ik_s \\ ik_r & ik_r & i\tilde{u}_o\cdot k\delta_{rs} \end{pmatrix} \begin{pmatrix} \tilde{\rho} \\ \tilde{T} \\ \tilde{u}_s \end{pmatrix} = \begin{pmatrix} \tilde{\sigma}_1 \\ \tilde{\sigma}_2 \\ \tilde{\sigma}_3 \end{pmatrix}$$

This is the system studied by H. Grad in his report [6]. It is simple to see by direct Fourier inversion, that the system (3.25), is just the linearized Euler equations with source terms. The results that Grad derived are essentially those usually gotten from inviscid flow theory. As is well-known from gas dynamics, the solution of equation (3.25) is singular along the Mach cone for supersonic flow (see Sec. 4). H. Grad [6], also showed the existence of a singularity along the flow axis behind the origin independently of Mach number. The divergence arises from the fact that Grad solved equations (3.25), whereas the comparable potential equation is solved in gas dynamics. The additional singularity arises by not excluding rotational flow (as is implicit in the potential equation), and is the inviscid of limit of the wake behind the body. Grad's solution is the correct fundamental solution of the Euler equations without a mass source.

4. Solutions to the Euler Approximation.

Before going on to the higher approximations, we work out the two dimensional analogue to Grad's analysis.

In order to carry out the inversion necessary for the solution of (3.25), we must first consider the source terms which occur on the right hand side of the equation. As was shown by H. Grad in [6], the Fourier inversion of a term such as σ_1 appears as an object of size O(L) located at the origin. Physically this means that the source appears smeared out to the extent of a mean free path. This smeared source can also be expanded as was the integral term in equation (3.17). Actually, this operation can be carried out earlier by expanding σ in the same way as indicated in (3.24).

This leads to an asymptotic representation, in the sense of distributions, and is just a multipole expansion in terms of the Dirac δ-function and its derivatives. Therefore, we have to the lowest order

$$(4.1)\quad \begin{cases} \sigma_1 \sim \int \tilde{\sigma} d\xi \\ \sigma_2 \sim \int (\frac{1}{3}\tilde{c}^2 - 1)\tilde{\sigma} d\tilde{\xi} \\ \sigma_3 \sim \int \tilde{c}\tilde{\sigma} d\tilde{\xi} \end{cases}$$

For the sake of simplicity we take

$$(4.2)\quad \int \tilde{\sigma}(\xi) d\tilde{\xi} = 0$$

which is a statement of mass conservation, and

$$(4.3)\quad \int \tilde{\sigma}(\xi)\tilde{\xi}^2 d\tilde{\xi} = 0$$

which states that no net energy exchange is taking place between the body and the fluid. We shall later show, in Section 9, the effect of heat addition. With these conditions, we take

$$(4.4)\quad \begin{cases} \sigma_3 \sim \tau u_0 \\ \sigma_2 \sim -\frac{2}{3}\tau u_0^2 \end{cases}$$

Therefore, τ is a negative constant representative of the drag, and is of the dimensions of length. These conservation statements are only sensible in talking about the asymptotic flow past a body. In talking about the fundamental solution we can still make the statement about the conservation of mass, as long as we agree not to consider flows with mass addition. On the other hand we no longer have the right to avail ourselves of the energy equilibrium equation (4.3), and the exclusion of lift, as is implicit in equation (4.4). These are in general needed when one attempts to actually construct solutions. We will return to these remarks in Section 9. For the present we assume equations (4.3) and (4.4) for the sake of simplicity.

We can simplify our calculation by noticing the 4×4 matrix (3.25) can be reduced to a 3×3 matrix, if we solve for $\tilde{\rho}$, $\tilde{T}$, and $k \cdot \tilde{u}$. This is accomplished by taking the inner product of the last row of equation (3.25) with k. In this case (3.25) becomes

$$(4.5) \qquad \begin{pmatrix} i(\tilde{u}_o \cdot k) & 0 & i \\ 0 & i(\tilde{u}_o \cdot k) & \frac{2}{3} i \\ ik^2 & ik^2 & i(\tilde{u}_o \cdot k) \end{pmatrix} \begin{pmatrix} \tilde{\rho} \\ \tilde{T} \\ k \cdot \tilde{u} \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{2}{3} \tau \tilde{u}_o{}^2 \\ \tau k \cdot \tilde{u}_o \end{pmatrix}$$

and inverting the matrix

$$(4.6) \qquad \begin{pmatrix} \tilde{\rho} \\ \tilde{T} \\ k \cdot \tilde{u} \end{pmatrix} = \frac{1}{D} \begin{pmatrix} \frac{2}{3}k^2 - (\tilde{u}_o \cdot k)^2 & -k^2 & \tilde{u}_o \cdot k \\ -\frac{2}{3} k^2 & k^2 - (\tilde{u}_o \cdot k)^2 & \frac{2}{3} \tilde{u}_o \cdot k \\ (\tilde{u}_o \cdot k)k^2 & (\tilde{u}_o \cdot k)k^2 & -(\tilde{u}_o \cdot k) \end{pmatrix} \begin{pmatrix} 0 \\ -\frac{2}{3} \tau \tilde{u}_o{}^2 \\ \tau \tilde{u}_o \cdot k \end{pmatrix}$$

with

(4.7) $$D = i(\tilde{u}_o \cdot k)[\tfrac{5}{3} k^2 - (\tilde{u}_o \cdot k)^2]$$

$$= i\tilde{u}_o k_1 \, [(\tfrac{5}{3} - u_o^2)k_1^2 + \tfrac{5}{3} k_2^2]$$

where the upstream velocity, $\tilde{u}_o$, is taken as oriented along the x_1-axis. Since

(4.8) $$\tilde{u}_o^2 = \tfrac{5}{3} M^2$$

where M is the Mach number, the coefficient of k_1^2 in D is proportional to $(1-M^2)$. Because the character of the equations changes from elliptic to hyperbolic as M increases through the value one, we consider each regime separately.

Supersonic Case

(4.9) $$(1 - \tfrac{3}{5} u_o^2) = \frac{-1}{\beta^2} \quad , \ \beta^2 > 0.$$

Solving for the density from the matrix equation (4.6)

(4.10) $$\tilde{\rho} = \tau \tilde{u}_o^2 \, (\tfrac{5}{3} k_1^2 + \tfrac{2}{3} k_2^2) \, \frac{1}{D}$$

and taking a Fourier inversion

(4.11) $$\tilde{\rho} = \frac{-3i \ \tau \tilde{u}_o \beta^2}{5} \left(\frac{5}{3} \frac{\partial^2}{\partial x^2} + \right.$$

$$\left. + \frac{2}{3} \frac{\partial^2}{\partial y^2}\right) \frac{1}{(2\pi)^2} \int \frac{e^{ik_1x+ik_2y} \ \overline{dk}}{k_1(k_1-\beta k_2)(k_1+\beta k_2)} \ .$$

Similarly we get for the temperature $\tilde{T}$

$$(4.12)\qquad \tilde{T} = \frac{2i\ \tau\tilde{u}_o\beta^2}{5}\left[-\,u_o{}^2\frac{\partial^2}{\partial x^2} + \right.$$

$$\left. + \frac{\partial^2}{\partial y^2}\right]\frac{1}{(2\pi)^2}\int \frac{e^{ik_1x+ik_2y}\ \overline{dk}}{k_1(k_1-\beta k_2)(k_1+\beta k_2)}\ .$$

Finally, in order to determine the velocity field, we return to (3.25) to get

$$(4.13)\qquad u_r = \frac{-k_r(\rho + T)}{\tilde{u}_o k_1} - \frac{i\tau\hat{\tilde{u}}_o}{k_1}\ ,$$

with $\hat{\tilde{u}}_o$, a unit vector. Inversion gives

$$(4.14)\qquad \tilde{u}_r = \frac{+i\tau(5\beta^2+2)}{3}\nabla_r\frac{\partial}{\partial x}\frac{1}{(2\pi)^2}\int\frac{e^{ik_1x+ik_2y}\ \overline{dk}}{k_1(k_1-\beta k_2)(k_1+\beta k_2)} -$$

$$-\ \tau i\,\frac{\hat{\tilde{u}}_o}{(2\pi)^2}\int\frac{e^{ik_1x+ik_2y}\ \overline{dk}}{k_1}\ .$$

In the evaluation of the two integrals which occur, the path of integration in the complex k_1-plane is taken slightly below the real axis. This eliminates advanced solutions, i.e., solutions propagating upstream. The integrations are easily carried out, in the sense of distributions, see Lighthill [17], and we get

$$(4.15)\qquad \frac{1}{(2\pi)^2}\int\frac{e^{ik\cdot x}\ \overline{dx}}{k_1} = iH(x)\delta(y)$$

where H(x) is the Heaviside step-function, and δ(y) the Dirac delta-function. The second integral is

$$(4.16)\qquad \frac{1}{(2\pi)^2}\int \frac{e^{ik\cdot x}\,\overline{dk}}{k_1(k_1-\beta k_2)(k_1+\beta k_2)} = \frac{iH(x)}{2\beta^2}\left\{|y| - \frac{1}{2}\,|y+\beta x| - \frac{1}{2}\,|y-\beta x|\right\} .$$

On substitution into equations (4.11), (4.12) and (4.14) we get

$$(4.17)\qquad \tilde{\rho} = \frac{2}{5}\,\tau\tilde{u}_o\delta(y)H(x) - \frac{\tau H(x)}{10\,u_o\cos\theta}\,(2u_o^2+5)[\delta(y\cos\theta + x\sin\theta) + \delta(y\cos\theta - x\sin\theta)]$$

$$(4.18)\qquad \tilde{T} = -\frac{2}{5}\,\tau\tilde{u}_o\delta(y)H(x) - \frac{\tau H(x)}{15\,u_o\cos\theta}\,(2u_o^2+5)[\delta(y\cos\theta + x\sin\theta) + \delta(y\cos\theta - x\sin\theta)]$$

$$(4.19)\qquad \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\tau H(x)\delta(y) + \frac{\tau H(x)(2u_o^2+5)}{6u_o^{\,2}}\begin{pmatrix} \sec\theta \\ \csc\theta \end{pmatrix}\delta(y\cos\theta + x\sin\theta) + \frac{\tau H(x)(2u_o^2+5)}{6u_o^{\,2}}\begin{pmatrix} \sec\theta \\ -\csc\theta \end{pmatrix}\delta(y\cos\theta - x\sin\theta)$$

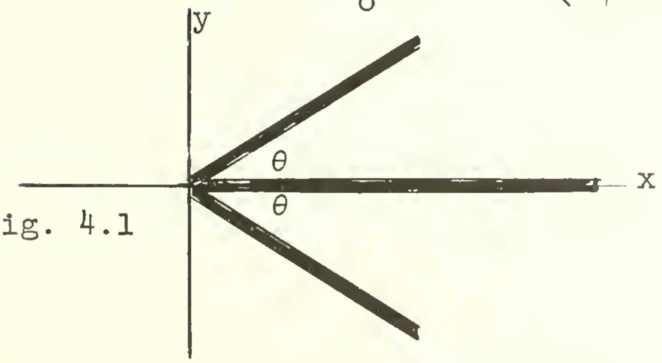

Fig. 4.1

The solution exhibited in equations (4.17-4.18) are non-zero only on the heavily marked lines in Figure 4.1. With the exception of the line concentration $\delta(y)H(x)$, the solutions are what we would anticipate from gas dynamic potential theory. A discussion of these singularities in the three dimensional can be found in [6], (the only qualitative difference in the solutions being that the three dimensional supersonic case exhibits debris behind the Mach cone, corresponding to the tail of the solution to the wave equation in two space dimensions). It suffices here to say that, when the supersonic flow past an object is viewed in the inviscid limit, the viscous layer, i.e., the Mach lines and the wake, are compressed into line discontinuities. It is one of the goals of this report to examine these regions.

Subsonic Flow: $(1 - \frac{3}{5}\tilde{u}_o^2) = \frac{1}{\alpha^2}$, $\alpha^2 > 0.$

The subsonic case is found in essentially the same way as the supersonic case. Rather than repeat the analysis, we just give the salient forms which appear.

After solving for the transformed flow field and inverting the Fourier transforms, we get

$$\left\{ \tilde{\rho} = \frac{3i\tau\tilde{u}_o}{5}\left(\frac{5}{3}\frac{\partial^2}{\partial x^2} + \frac{2}{3}\frac{\partial^2}{\partial y^2}\right)\frac{\alpha^2}{(2\pi)^2}\int \frac{e^{ik_1x+ik_2y}\,\overline{dk}}{k_1(k_1-i\alpha k_2)(k_1+i\alpha k_2)} \right.$$

$$(4.20)\qquad \left\{\begin{aligned} \tilde{T} &= \frac{3i\tau\tilde{u}_o}{5}\left(u_o^2\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)\frac{\alpha^2}{(2\pi)^2}\int\frac{e^{ik_1x+ik_2y}\,\overline{dk}}{k_1(k_1-i\alpha k_2)(k_1+i\alpha k_2)} \\ \begin{pmatrix}\tilde{u}_1\\ \tilde{u}_2\end{pmatrix} &= \tau\,\frac{(5\alpha^2-2)}{3}\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{pmatrix}\frac{1}{(2\pi)^2}\int\frac{e^{ik_1x+ik_2y}\,\overline{dk}}{(k_1-i\alpha k_2)(k_1+i\alpha k_2)} - \\ &\qquad - i\tau\begin{pmatrix}1\\ 0\end{pmatrix}\frac{1}{(2\pi)^2}\int\frac{e^{ik_1x+ik_2y}\,\overline{dk}}{k_1} \end{aligned}\right.$$

As before the integrals, in the ordinary sense, are divergent, but exist in the sense of distributions, [17].

$$(4.21)\qquad \frac{1}{(2\pi)^2}\int\frac{e^{ik_1x\,+\,ik_2y}\,\overline{dk}}{k_1} = i\delta(y)H(x)$$

$$(4.22)\qquad \frac{1}{(2\pi)^2}\int\frac{e^{ik_1x\,+\,ik_2y}\,\overline{dk}}{(k_1-i\alpha k_2)(k_1+i\alpha k_2)} = -\frac{1}{\alpha 2\pi}\log r$$

$$(4.23)\qquad \frac{\alpha^2}{(2\pi)^2}\int\frac{e^{ik_1x\,+\,ik_2y}\,\overline{dk}}{k_1(k_1-i\alpha k_2)(k_1+i\alpha k_2)} = -\frac{i|y|H(x)}{2} - \\ -\frac{i\alpha x}{2\pi}\log r+\frac{iy}{2\pi}\tan^{-1}\left(\frac{y}{\alpha x}\right)$$

with

$$(4.24)\qquad r = \sqrt{\alpha^2x^2+y^2}\ .$$

Inserting these into the flow field (4.20), we have

$$(4.25)\quad \begin{cases} \tilde{\rho} = \dfrac{2\tau\tilde{u}_o}{5} H(x)\delta(y) + \dfrac{\tau\tilde{u}_o}{10} \dfrac{\alpha x}{\pi r^2} [5\alpha^2 - 2] \\ \tilde{T} = - \dfrac{2\tau u_o}{5} \delta(y)H(x) + \dfrac{2}{3} \dfrac{\alpha x \tau u_o}{\pi r^2} [5\alpha^2 - 2] \\ \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tau\delta(y)H(x) - \dfrac{\tau(5\alpha^2 - 2)}{6\pi\alpha} \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{pmatrix} \log r \end{cases}$$

The subsonic case given above is listed only for completeness. With the exception of the wake and those high subsonic velocities which may be considered transonic, it will not be studied.

Sonic Flow: $u_o^2 = \frac{5}{3}$

Although the linearized inviscid equations give seemingly absurd results for just sonic flow, it is of interest for comparison with later work to have this computation. Returning to equations (4.11), (4.12) and (4.14) we have in this case:

$$(4.26)\qquad \tilde{\rho} = \frac{i\tau}{u_o} \left(\frac{5}{3} \frac{\partial^2}{\partial x^2} + \frac{2}{3} \frac{\partial^2}{\partial y^2}\right) \frac{1}{(2\pi)^2} \int \frac{e^{ik_1x + ik_2y}\, \overline{dk}}{k_1 k_2^2}$$

$$(4.27)\qquad \tilde{T} = \frac{2}{3} \frac{i\tau}{u_o} \left(\frac{5}{3} \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) \frac{1}{(2\pi)^2} \int \frac{e^{ik_1x + ik_2y}\, \overline{dk}}{k_1 k_2^2}$$

$$(4.28)\qquad \tilde{u}_r = \frac{\tau 5}{3} \nabla_r \frac{1}{(2\pi)^2} \int \frac{e^{ik_1x + ik_2y}\, \overline{dk}}{k_2^2} -$$

$$- \frac{i\tau\hat{\tilde{u}}_o}{(2\pi)^2} \int \frac{e^{ikx + iky}\, \overline{dk}}{k_1}.$$

The integrations which occur are straightforward, as before.

$$(4.29) \qquad \frac{1}{(2\pi)^2}\int \frac{e^{ik_1x + ik_2y}\,\overline{dk}}{k_1k_2^{\,2}} = -iH(x)\,\frac{|y|}{2}$$

$$(4.30) \qquad \frac{1}{(2\pi)^2}\int \frac{e^{ik_1x + ik_2y}\,\overline{dk}}{k_2^{\,2}} = -\,\delta(x)\,\frac{|y|}{2}$$

Substituting into equations (4.22-4.24) we get

$$(4.31) \qquad \begin{cases} \tilde{\rho} = \dfrac{\tau}{2\tilde{u}_o}\left(\dfrac{5}{3}\,\delta'(x)|y| + \dfrac{4}{3}\,H(x)\delta(y)\right) \\[2ex] \tilde{T} = \dfrac{\tau}{3\tilde{u}_o}\left(\dfrac{5}{3}\,\delta'(x)|y| - \dfrac{4}{3}\,H(x)\delta(y)\right) \\[2ex] \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix} = \dfrac{-5\tau}{3}\begin{pmatrix} \dfrac{\partial}{\partial x} \\[1ex] \dfrac{\partial}{\partial y} \end{pmatrix}\dfrac{\delta(x)\,|y|}{2} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\tau H(x)\delta(y) \end{cases}$$

This solution grows linearily in the y-direction. Because of this, the assumption of linearization soon fails and in addition it is simple to show that an infinite drag results. Nevertheless, these solutions become meaningful when we examine their viscous counterpart in Section 8.

5. The Higher Approximations.

In this section we consider the system of equations for $\tilde{\rho}$, $\tilde{T}$, $\tilde{u}$, when we take the next approximation to Ω, defined by equation (3.22),

$$(5.1) \qquad \Omega = \omega\left\{1 - iLk\cdot\tilde{\xi} + (iLk\cdot\tilde{\xi})^2\right\}$$

$$= \omega\left\{1 - iLk\cdot\tilde{c} - iL(k\cdot\tilde{u}_o) - L^2(k\cdot\tilde{c})^2 - L^2(k\cdot\tilde{c})k\cdot\tilde{u}_o - \right.$$

$$\left. - L^2(k\cdot\tilde{u}_o)^2\right\} \ .$$

When this is substituted into the matrix equation (3.21) we get a linear, inhomogeneous matrix system, as before, in $\tilde{\rho}$, $\tilde{T}$, $\tilde{u}$.

$$(5.2) \quad \begin{pmatrix} i\tilde{u}_o\cdot k+L(\tilde{u}_o\cdot k)^2 + Lk^2 & Lk^2 & ik_s+2L\tilde{u}_o\cdot kk_s \\ \frac{2}{3}\,Lk^2 & i(\tilde{u}_o\cdot k)+L(\tilde{u}_o\cdot k)^2 + \frac{7}{3}\,Lk^2 & \frac{2}{3}\,(ik_s + 2L\tilde{u}_o\cdot kk_s) \\ ik_r+2L(\tilde{u}_o\cdot k)k_r & ik_r+2L(\tilde{u}_o\cdot k)k_r & [i\tilde{u}_o\cdot k+(\tilde{u}_o\cdot k)^2L + k^2L]\delta_{rs}+2Lk_rk_s \end{pmatrix} \begin{pmatrix} \tilde{\rho} \\ \tilde{T} \\ \tilde{u}_s \end{pmatrix}$$

$$= \begin{pmatrix} \tilde{\sigma}_1 \\ \tilde{\sigma}_2 \\ \tilde{\sigma}_3 \end{pmatrix}$$

The integrations involved in determining the matrix elements are well-known and a complete list is given in [12]. In this form the equations of (5.2) do not seem to bear any relation to known hydrodynamic systems; however, this is just midleading. The system is the Navier-Stokes set linearized away from the constant free stream velocity. This is usually referred to as the Oseen equations. For instance, on multiplying the system (5.2) by

$$(5.3)\quad \begin{pmatrix} 1 + iL(u_o \cdot k) & 0 & iLk_r \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

we get

$$(5.4)\begin{pmatrix} i(\tilde{u}_o \cdot k) & 0 & ik_s \\ \frac{2}{3}Lk^2 & \begin{matrix} i(\tilde{u}_o \cdot k)+L(\tilde{u}_o \cdot k)^2 \\ +\frac{7}{3}Lk^2 \end{matrix} & \frac{2}{3}(i+2L\tilde{u}_o \cdot k)k_s \\ \left\{i+2L(\tilde{u}_o \cdot k)\right\}k_r & \left\{i+2L(u_o \cdot k)\right\}k_r & \begin{matrix} \{iu_o \cdot k+(\tilde{u}_o \cdot k)^2L+ \\ +k^2L\}\delta_{rs}+2Lk_rk_s \end{matrix} \end{pmatrix}\begin{pmatrix} \tilde{\rho} \\ \tilde{T} \\ \dot{u}_s \end{pmatrix}$$

$$= \begin{pmatrix} O(L^2) \\ \tilde{\sigma}_2 \\ \tilde{\sigma}_3 \end{pmatrix}$$

The source relations of Section 5 have been applied to the first row. From the first row, we see that linearized continuity equation is satisfied to $O(L^2)$. The linearized continuity equation is in fact satisfied identically, as can be seen by merely integrating equation (3.2), and then linearizing. To get the familiar Navier-Stokes system we have only to multiply (5.2) by the matrix

$$(5.5) \quad \begin{pmatrix} 1 + iL(\tilde{u}_o \cdot k) & 0 & iLk_r \\ 0 & 1+L(\tilde{u}_o \cdot k)i & \frac{2}{3}\, iLk_r \\ iLk_r & iLk_r & 1+iL(\tilde{u}_o \cdot k) \end{pmatrix}$$

However, since there is no compelling reason to use the Navier-Stokes system, we will take (5.4) as our system.

That we arrive at the Navier-Stokes set should cause us no great surprise. Without going into details, it is simple to see that the expansion of Ω in the above analysis is identical to an iterative prodecure. This mimics the Chapman-Enskog method in the first two steps (the first two steps in the Chapman-Enskog method lead to the Navier-Stokes set, see [16]).

Since we have a system equivalent to the Navier-Stokes set, we have also a calculation of the transport coefficients. These will, of course, depend upon the choice of ν. A systematic discussion of the Boltzmann equation leading to the single relaxation model, is given by Gross and Jackson [4]. The value of ν that the analysis indicates, results in a viscosity coefficient which Chapman and Cowling call the first approximation to the viscosity coefficient. However, the heat conductivity is off by a factor of 3/2. This fact has already been noticed by Gross [3].

6. Boundary Layers.

Starting with the matrix equation (5.4), one can in principle solve for the flow field (ρ, T, u), (indeed even starting with (3.21) the method of solution is clear). One has only to invert the matrix equation, to solve for the Fourier transformed field, and then invert the transforms. Formally

$$(6.1)\qquad \begin{pmatrix} \tilde{\rho} \\ \tilde{T} \\ \tilde{u} \end{pmatrix} = \frac{1}{(2\pi)^n} \int e^{ik\cdot x} \, \frac{\bar{\bar{C}}}{D} \begin{pmatrix} 0 \\ \tilde{\sigma}_2 \\ \tilde{\sigma}_1 \end{pmatrix} \bar{\partial} k$$

where n is the dimension of the space, D the determinant of the matrix, and $\bar{\bar{C}}$ the transpose of the cofactor matrix. However, such a plan does not seem feasible. Instead we adopt the philosophy, that the inviscid solution developed in Section 3 is adequate except in the neighborhood of the singular lines. That is, when the solution of the inviscid equations has meaning as a function in the ordinary sense it is adequate, and when the solution has meaning only in the sense of distributions it points out an inadequacy in the solution, and hence in the governing equation in that region of space. In the remainder of this section, we consider the form of the second approximation, Navier-Stokes, in the regions of singularity of the first approximation, Euler.

As in the Euler approximation we can separate our system so that we only consider ρ, T, and $k\cdot u$. Taking the divergence of the last row of (5.4), the matrix equation becomes

$$(6.2)\quad \begin{pmatrix} i(\tilde{u}_o\cdot k) & 0 & i \\ \frac{2}{3}Lk^2 & \begin{matrix} i(\tilde{u}_o\cdot k)+L(\tilde{u}_o\cdot k)^2 \\ +\frac{7}{3}Lk^2 \end{matrix} & \frac{2}{3}(i+2L\tilde{u}_o\cdot k) \\ \{i+2L\tilde{u}_o\cdot k\}k^2 & \{i+2L(\tilde{u}_o\cdot k)\}k^2 & \begin{matrix} i\tilde{u}_o\cdot k+(\tilde{u}_o\cdot k)^2L \\ +3k^2L \end{matrix} \end{pmatrix} \begin{pmatrix} \tilde{\rho} \\ \tilde{T} \\ k\cdot\tilde{u} \end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{\sigma}_2 \\ k\cdot\tilde{\sigma}_3 \end{pmatrix}$$

The determinant of the matrix is easily computed, and to O(L) is given by

$$(6.3)\qquad D = -i(\tilde{u}_o\cdot k)^3 + i\,\frac{5}{3}(\tilde{u}_o\cdot k)k^2 + \frac{5}{3}k^4L - 2(\tilde{u}_o\cdot k)^4L + \\ + \frac{1}{3}(\tilde{u}_o\cdot k)^2k^2L.$$

As can be seen from (6.1), D, if Fourier inverted, i.e.,

$$(6.4) \qquad D = (\tilde{u}_o\cdot\nabla)^3 - \frac{5}{3}(\tilde{u}_o\cdot\nabla)\nabla^2 + \frac{5}{3}L\nabla^4 - 2L(\tilde{u}_o\cdot\nabla)^4 + \frac{L}{3}(\tilde{u}_o\cdot\nabla)^2\nabla^2,$$

is the differential operator governing each of the dependent variables $\tilde{\rho}$, $\tilde{T}$, $\nabla\cdot u$. Therefore, we can expect to find the correct approximations in the neighbpthood of the singularities of the Euler system, by examining the operator D.

Henceforth we restrict attention to the two-dimensional case, and we will introduce pertinent results of the three-dimensional case only for the sake of comparison. Let us first consider the neighborhood of the Mach line as in Figure 4.1. First we rotate our coordinate frame through the Mach angle θ, as in Figure 6.1, so that h denotes distance along the Mach line, and t the coordinate normal to the Mach line.

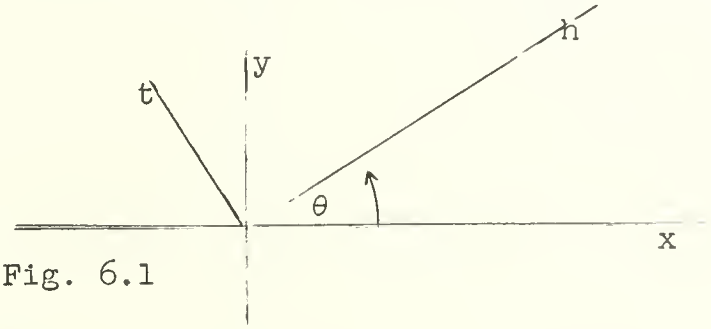

Fig. 6.1

We assume that the Mach line singularity is actually a narrow layer through which the physical variables change, and we spread the layer in the lateral direction by factor 1/δ, where δ is the layer thickness.

(6.5) $\quad t/\delta = t'$.

This should actually be thought of as $\frac{t/R}{\lambda/R}$, $\delta = \lambda/R$, where R is the large distance from the object mentioned in Section 3, and λ is the actual thickness of the layer. Equivalent to the spreading is the statement that

(6.6) $\quad \partial/\partial t \sim O(1/\delta)$,

or if we deal with wave number space, that the wave number corresponding to t is $O(1/\delta)$. We now carry out the analysis in wave number space, i.e., we consider the determinant (6.3). Denoting the wave number vector $\overline{k}$ by

(6.7) $\quad \overline{k} = (k_1, k_2)$

we perform the rotation indicated in Figure 6.1, and represent it by

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} \tag{6.8}$$

so that m denotes the wave number across the layer, and n the wave number along the layer. Substituting the transformation (6.8) in the determinant (6.3) we have

$$D = -\frac{5}{3}\, i\tilde{u}_0 mn^2 \sin\theta - \frac{10}{3}\, i\tilde{u}_0 m^2 n \cos\theta - \frac{10}{3}\, Lm^4 + \tag{6.9}$$
$$+ O(Lm^3 n,\ Lm^2 n^2,\ Lmn^3,\ Ln^4,\ n^3).$$

Use has been made of the relation

(6.10) $\quad \tilde{u}_o^{\,2} \sin^2\theta = 5/3.$

Details of the remainder terms in (6.9) are not given since they are negligible compared to the terms carried.

Let us assume that θ is neither near zero nor $\pi/2$; then $\cos\theta$, $\sin\theta \sim O(1)$. Then according to (6.6)

(6.11) $\quad -\frac{5}{3} Lu_o mn^2 \sin\theta \sim O(1/\delta)$

(6.12) $\quad -\frac{10}{3} Lu_o m^2 n \cos\theta \sim O(1/\delta^2)$

(6.13) $\quad -\frac{10}{3} Lm^4 \sim O(L/\delta^4).$

Since δ is considered to be a small parameter, these relations indicate that (6.12) can be neglected relative to (6.11). Furthermore, since the Euler term is inadequate for describing the flow in the region of interest, we take (6.11) and (6.13) to be the same order of magnitude. We conclude therefore that

(6.14) $\quad D \sim -\frac{10}{3} i\tilde{u}_o m^2 n \cos\theta - \frac{10}{3} Lm^4.$

Also since $O(L/\delta^4) = O(1/\delta^2)$ we can further conclude that the thickness of the layer, λ, varies as $\sim (RL)^{1/2}$. Furthermore, from (6.14) we see that the layer is governed by a diffusion phenomenon.

As a partial justification of the procedure (fuller justification comes when we find the solution is in agreement with the assumptions made), we should verify that the next order terms, i.e., L^2 terms, are ignorable. Terms of $O(L^2)$ arise in two different ways, first those terms which we have been neglecting in expanding the determinant of the matrix in equation (6.2). By inspection these are $O(L^2/\delta^5)$. Second, those which arise in carrying the expansion at Ω one step further, that is, taking

$$(6.15) \qquad \Omega \sim 1 - (iLk\cdot\xi) + (iLk\cdot\xi)^2 - (iLk\cdot\xi)^3.$$

We shall refer to the system of equations implied by (6.15) as "Burnett", since they are in a formal sense similar to the Burnett equations. Considering this system we again find that the set can be separated, and further that the largest order contribution to the determinant is $O(L^2/\delta^5)$.

We now have three types of terms, which we refer to as Euler, Navier-Stokes and Burnett; and which we estimate to be $O(1/\delta^2)$, $O(L/\delta^4)$, $O(L^2/\delta^5)$, respectively. Assuming that the Euler and Navier-Stokes terms are of the same order, leads to a $\delta \sim O(L^{1/2})$. If this is substituted into the Burnett term, $O(L^2/\delta^5)$, it is clear that the Burnett term is negligible under this assumption. The assumption that the Euler and Navier-Stokes terms are of the same order will in a sense be removed. It will now be shown to be the only possibility.

As we have mentioned before, we expect the solution to the Euler equations to give an adequate description of the flow in the greater part of the field. Only in certain narrow regions do the Euler solutions break down. As we enter these "narrow" regions from the Euler region, we expect on physical grounds (and assume mathematically) that the differential operator governing this region changes continuously. In other words, there are differential terms, previously neglected, which gradually become as important as the Euler terms. There is no sudden change of one operator into another. The changing of the Euler operator into the Navier-Stokes operator has been shown in the last paragraph to be consistent. If we attempt to change the Euler operator, $O(1/\delta^2)$, into the Burnett operator, $O(L^2/\delta^5)$, we find that $\delta \sim O(L^{2/3})$. This is contradictory since it implies that the Navier-Stokes operator is of a lower order.

We should not discount the possibility that after the Euler operator has changed into the Navier-Stokes operator, that the Navier-Stokes operator will change to Burnett. Equating the orders of the Burnett and Navier-Stokes term, we find that such a transition can indeed take place (the Euler terms are of higher order), but in a sublayer of actual thickness $\delta = O(L)$. It seems likely however that in such a region the remainder terms, i.e., the higher order terms of the expansion in L, would be equally important. In other words, the detail which the Burnett equations try to furnish in the sublayer can only be gotten from the full kinetic equation itself.

So far we have discussed in detail the approximations involved in the purely supersonic case; the same arguments apply in the other ranges. In the almost sonic region, $|\cos\theta| << O(\delta)$, when the upstream velocity varies from high subsonic to low supersonic speeds, the operator (6.9) is approximately given by

$$(6.16) \qquad D \sim -\frac{5}{3} i\tilde{u}_o mn^2 - \frac{10}{3} Lm^2.$$

All arguments presented above for the purely supersonic case apply again. Here, however, the thickness of the layer varies as $\sim (R^2L)^{1/3}$. In the event that $\cos\theta \sim O(\delta)$, it is clear that all three terms occuring in (6.9) are of equal importance. Therefore, in order to consider a solution which is uniformly accurate with respect to Mach number in all the regions, we must consider the entire operator D, less the remainder term, i.e., we take

$$(6.17) \qquad D = -\frac{5}{3} i\tilde{u}_o mn^2 \sin\theta - \frac{10}{3} i\tilde{u}_o m^2 n \cos\theta - \frac{10}{3} Lm^4.$$

The corresponding differential operator is

$$(6.18) \qquad D = \frac{5}{3}\frac{\partial}{\partial t}\left(\tilde{u}_o \sin\theta \frac{\partial^2}{\partial h^2} + 2\tilde{u}_o \cos\theta \frac{\partial^2}{\partial h \partial t} - 2L\frac{\partial^3}{\partial t^3}\right).$$

In the supersonic case the differential operator becomes

$$(6.19) \qquad D \sim \frac{10}{3}\frac{\partial^2}{\partial t^2}\left(\tilde{u}_o \cos\theta \frac{\partial}{\partial h} - L\frac{\partial^2}{\partial t^2}\right)$$

and as indicated before the layer spreading is governed by a diffusion equation. In this case we expect the solution to depend on the similarity variable $(t^2 u \cos\theta/Lh)$.

In the almost sonic case ($|\cos\theta| << \delta$), the differential operator becomes

$$(6.20) \qquad D \sim \frac{5}{3}\frac{\partial}{\partial t}\left(\tilde{u}_o \frac{\partial^2}{\partial h^2} - 2L\frac{\partial^3}{\partial t^3}\right)$$

(at this point y, -x could replace h, t). The operator in the parenthesis of equation (6.20) does not seem to have been studied. We can expect the solution to depend on the similarity variable $t^3 u_o/h^2 L$). The more general case given by the operator in (6.18) will therefore give a transition from one similarity variable $(t^2 u_o \cos\theta/Lh)$, to the other $(t^3 u_o/Lh^2)$, as we move from supersonic flow to high subsonic flow.

There is still another boundary layer to be considered. Looking at Figure 4.1, we see a singularity $H(y)\delta(x)$ which corresponds to the wake behind the object. The three-dimensional wake has already been considered by H. Grad [13]. To consider the wake we have to "spread" the region along the positive x-axis. Carrying on the same analysis as used in the above cases we see that we can estimate the determinant D, (6.2) by

$$(6.21) \qquad D \sim i\,\frac{5}{3}\,\tilde{u}_o k_1 k_2{}^2 + \frac{5}{3}\,k_2{}^4 L.$$

Here, $\delta = O(L^{1/2})$ and as in the supersonic Mach layer, the thickness of the wake grows as $(RL)^{1/2}$. Corresponding to the estimate of D, (6.21) we have the differential operator

$$(6.22) \qquad D = -\frac{5}{3}\frac{\partial^2}{\partial y^2}\left(u_o \frac{\partial}{\partial x} - L\frac{\partial^2}{\partial y^2}\right)$$

which again is essentially the diffusion operator. And again we can expect to find the similarity variable (y^2/hx) to occur in the solution for the flow field.

In our discussion, we have ignored the case when the Mach angle is small. In this case the free stream velocity, $\tilde{u}_o$, becomes very large. This is the hypersonic limit, and we do not consider it in this report.

7. Wake Layer.

In this section we apply the approximations developed in Section 6, to our matrix system (5.4). Recalling that for the wake we had

$$(7.1) \qquad D = i\,\frac{5}{3}\,\tilde{u}_o k_1 k_2^{\,2} + \frac{5}{3}\,k_2^{\,4} L$$

with

$$(7.2) \qquad \delta = O(L^{1/2})$$

we can easily compute to lowest order terms, the inverse matrix of system (5.4). Performing the matrix inversion we get,

$$(7.3)\quad \begin{pmatrix} \tilde{\rho} \\ \tilde{T} \\ k\cdot\tilde{u} \end{pmatrix} = \frac{1}{D}\begin{pmatrix} \frac{2}{3}k_1{}^2 & -k_2{}^2 & \tilde{u}_o k_1 - \frac{7}{3} iLk_2^2 \\ -\frac{2}{3}k_2{}^2 & k_2{}^2 & \frac{2}{3}(\tilde{u}_o k_1 + iLk_2^2) \\ -\frac{5}{3}k_2^4 iL & \tilde{u}_o k_1 k_2^2 & (\tilde{u}_o^2 k_1^2 + \frac{7}{3} i\tilde{u}_o L k_1 k_2^2) \end{pmatrix} \begin{pmatrix} 0 \\ -\frac{2}{3}\tau\tilde{u}_o^2 \\ \tau\tilde{u}_o k_1 \end{pmatrix}$$

We note immediately from the matrix in (7.3) that the last column is of a higher order than the second column. Therefore, the solution for $\tilde{\rho}$, $\tilde{T}$, and $\nabla\cdot\tilde{u}$ is to lowest order terms, only dependent on the entropy source. To this order then,

$$(7.4)\qquad \tilde{\rho} = -\frac{2}{5}\,\frac{iu_o\tau}{k_1 - ik_2^2L/u_o}$$

and on inverting the Fourier transform, the density is

$$(7.5)\qquad \tilde{\rho} = -\frac{2iu_o\tau}{(2\pi)^2}\int \frac{e^{ik\cdot x}\,\overline{dk}}{k_1 - ik_2^2L/u_o}\ .$$

By inspection of the system we have immediately

$$(7.6)\qquad \tilde{T} = -\tilde{\rho},$$

so that the perturbed pressure vanishes to this order. Lastly from (7.3)

$$(7.7)\qquad \kappa\cdot\tilde{u} = \frac{2}{5}\, i\tilde{u}_o^2\tau\,\frac{k_1}{k_1 - ik_2^2L/u_o}\ .$$

In order to determine the velocity field, we must return to the original system of equations (5.4), from which we get to lowest order terms

$$(7.8)\qquad (i\tilde{u}_o k_1 + k_2^2 L)\begin{pmatrix}\tilde{u}_1\\ \tilde{u}_2\end{pmatrix} = \tau\begin{pmatrix}\tilde{u}_o\\ 0\end{pmatrix} - 2\begin{pmatrix}k_1\\ k_2\end{pmatrix} k\cdot u\ L$$

and inverting

$$(7.9)\qquad \begin{pmatrix}\tilde{u}_1\\ \tilde{u}_2\end{pmatrix} = \frac{i\tau\begin{pmatrix}\tilde{u}_o\\ 0\end{pmatrix}}{(2\pi)^2}\int \frac{e^{ik\cdot x}\ \overline{dk}}{(k_1 - ik_2^2 L/u_o)} - \frac{L4u_o\tau}{5(2\pi)^2}\int \frac{e^{ik\cdot x}\begin{pmatrix}k_1\\ k_2\end{pmatrix}k_1\overline{dk}}{(k_1 - ik_2^2 L/u)^2}\ .$$

The flow field is therefore determined by the integrals

$$(7.10)\qquad I = \frac{1}{(2\pi)^2}\int \frac{e^{ik\cdot x}\ \overline{dk}}{k_1 - ik_2^2 L/\tilde{u}_o}$$

and

$$(7.11)\qquad II = \frac{1}{(2\pi)^2}\int \frac{e^{ik\cdot x}\qquad \overline{dk}}{(k_1 - ik_2^2 L/\tilde{u}_o)^2}$$

both of which are easily evaluated.

$$(7.12)\qquad I = \frac{iH(x)}{2}\left(\frac{u_o}{\pi Lx}\right)^{1/2} e^{-y^2\tilde{u}_o/4Lx}$$

$$(7.13)\qquad II = \frac{-xH(x)}{2}\left(\frac{u_o}{\pi Lx}\right)^{1/2} e^{-y^2\tilde{u}_o/4Lx}$$

Substituting the forms I and II into (7.5), (7.6), and (7.9) we determine the flow field

$$\tilde{\rho} = \frac{2}{5}\,\tilde{u}_o\tau\, H(x)\left(\frac{\tilde{u}_o}{\pi 4Lx}\right)^{1/2} e^{-y^2\tilde{u}_o/4Lx}$$

$$\tilde{T} = -\,\frac{2}{5}\,\tilde{u}_o\tau\, H(x)\left(\frac{\tilde{u}_o}{\pi 4Lx}\right)^{1/2} e^{-y^2\tilde{u}_o/4Lx}$$

$$(7.14)\qquad \tilde{u}_1 = \tau H(x)\left(\frac{\tilde{u}_o}{4\pi Lx}\right)^{1/2} e^{-y^2\tilde{u}_o/4Lx}$$

$$\tilde{u}_2 = \frac{4}{5}\,\tau u_o^2\, yH(x)\frac{\partial}{\partial x}\left[\left(\frac{u_o}{4\pi Lx}\right)^{1/2} e^{-y^2u_o/4Lx}\right]$$

We notice from the solution that (y^2/Lx) is indeed the similarity variable, and that the wake thickness is $O(\{Lx\}^{1/2})$.

At this point, it is important to make the distinction between solving for a fundamental solution, and solving for flow past an object. In the former case it is not necessary to demand that products of the solutions are small compared with respect to the solutions. On the other hand, the latter case absolutely demands that this be true. This is in order to agree with the linearization which we performed at the outset. For the case at hand, the linearization is valid when the quantity $\tau\tilde{u}_o^2/4\pi Lx$ is small compared with unity.

Regarding (7.14) as a fundamental solution, and taking the limit as L ⟶ 0, we recover the fundamental solutions of the Euler approximation (4.19-4.19). This follows immediately by noting that

$$(7.15)\qquad \lim_{L\to 0}\left(\frac{u_o}{4\pi Lx}\right)^{1/2} e^{-y^2\tilde{u}_o/4Lx} = \delta(y).$$

See for instance reference [17].

It is important to notice that the convergence to the Dirac δ-function is not uniform with respect to x. For any small $L > 0$, we can find a large enough x such that the wake is wide enough to invalidate an inviscid description. A similar effect occurs in the next section in connection with the Mach layer spread. We will further elaborate on this point there.

8. Mach Layer.

We now make use of the approximations developed in Section 6 for the layer in the neighborhood of the Mach lines. Recalling our calculations in Section 6, we found that in order to obtain a description, uniformly valid from supersonic to high subsonic speeds, we should take

$$D = -\frac{5}{3}\, i\tilde{u}_o m n^2 \sin\theta - \frac{10}{3}\, i\tilde{u}_o m^2 n \cos\theta - \frac{10}{3} L m^4 . \tag{8.1}$$

In addition it was found that δ varies from $O(L^{1/2})$ to $O(L^{1/3})$ as the flow goes from supersonic to subsonic speeds. Applying this to the Navier-Stokes approximation and carrying only lowest orders, we find, on inverting equation (6.2), that

$$(8.2)\quad \begin{pmatrix} \tilde{\rho} \\ \tilde{T} \\ k\cdot\tilde{u} \end{pmatrix} = \frac{1}{D} \begin{pmatrix} -m^2\sin^2\theta & -m^2 & -\tilde{u}_o m\sin\theta \\ -\frac{2}{3}m^2 & -\frac{2}{3}m^2 & -\frac{2}{3}\tilde{u}_o m\sin\theta \\ \tilde{u}_o m^3\sin\theta & -\tilde{u}_o m^3\sin\theta & -\frac{5}{3}m^2 \end{pmatrix} \begin{pmatrix} 0 \\ -\frac{2}{3}\tau\tilde{u}_o^2 \\ -\tilde{u}_o\tau m\sin\theta \end{pmatrix}$$

Solving for the transformed density to the lowest order,

$$(8.3)\quad \tilde{\rho} = \frac{i(2u_o^2+5)\tau}{5u_oL}\,\frac{m}{\left(\frac{n^2\sin\theta}{L} + \frac{2mn\cos\theta}{L} - \frac{2im^3}{u_o}\right)}$$

and then inverting to get the density, we have

$$(8.4)\quad \tilde{\rho} = \frac{i(2\tilde{u}_o^2+5)\tau}{5\sin\theta\, L}\,\frac{1}{(2\pi)^2u_o}\int \frac{e^{imt+inh}\, m\,dm\,dn}{\left(\frac{n^2}{L} + \frac{2mn\cot\theta}{L} - \frac{2im^3\csc\theta}{u_o}\right)} .$$

On inspection we see that

$$(8.5)\quad \tilde{T} = \frac{2}{3}\tilde{\rho}$$

so that to this order the flow is adiabatic. Also by inspection

$$(8.6)\quad k\cdot\tilde{u} = \tilde{u}_o m \text{ sim } \theta\, \rho .$$

As before, in order to get the expression for the velocity we have to return to the unseparated system (5.4), and from there we get

$$(8.7)\quad [i\tilde{u}_o\cdot k + (\tilde{u}_o\cdot k)^2L + k^2L]\begin{pmatrix}\tilde{u}_1\\ u_2\end{pmatrix} = \tau\begin{pmatrix}u_o\\ 0\end{pmatrix} - [(i+2L\tilde{u}_o\cdot k)\tilde{\rho} + 2Lk\cdot\tilde{u}]\begin{pmatrix}k_1\\ k_2\end{pmatrix}$$

Again only carrying lowest order terms we have

$$(8.8) \qquad (\tilde{u}_1, \tilde{u}_2) = \left(-\frac{1}{\tilde{u}_o}, \frac{m \cos \theta + n \sin \theta}{\tilde{u}_o m \sin \theta}\right) \frac{5}{3} \tilde{\rho} \, .$$

On taking a Fourier inversion of equation (8.8) we get

$$(8.9) \qquad \tilde{u}_1 = \frac{-i(2\tilde{u}_o^2+5)\tau}{3u_o \sin \theta \, L} \, \frac{1}{u_o(2\pi)^2} \int \frac{e^{imt+inh} \, m \, dm \, dn}{\left(\frac{n^2}{L} + \frac{2mn \cot \theta}{L} - \frac{2im^3 \csc \theta}{\tilde{u}_o}\right)}$$

and

$$(8.10) \qquad \tilde{u}_2 = \frac{iu_o(2\tilde{u}_o^2+5)\tau \cos \theta}{5L} \, \frac{1}{\tilde{u}_o(2\pi)^2} \int \frac{e^{imt + inh} \, m \, dm \, dn}{\left(\frac{n^2}{L} + \frac{2mn\cot \theta}{L} - \frac{2im^3 \csc \theta}{\tilde{u}_o}\right)}$$

$$+ \frac{i(2\tilde{u}_o^2+5)\tau}{3u_o \sin \theta \, L} \, \frac{1}{\tilde{u}_o(2\pi)^2} \int \frac{e^{imt + inh} \, n \, dm \, dn}{\left(\frac{n^2}{L} + \frac{2mn \cot \theta}{L} - \frac{2im^3 \csc \theta}{\tilde{u}_o}\right)} \, .$$

The integrals

$$(8.11) \qquad I = \frac{1}{\tilde{u}_o(2\pi)^2} \int \frac{e^{imt + inh} \, m \, dm \, dn}{\left(\frac{n^2}{L} + \frac{2mn \cot \theta}{L} - \frac{2im^3 \csc \theta}{\tilde{u}_o}\right)}$$

and

$$(8.12) \qquad II = \frac{1}{\tilde{u}_o(2\pi)^2} \int \frac{e^{imt + inh} \, n \, dm \, dn}{\left(\frac{n^2}{L} + \frac{2mn \cot \theta}{L} - \frac{2im^3 \csc \theta}{\tilde{u}_o}\right)}$$

are evaluated in Appendix A, in closed form in terms of an Airy integral. Therefore with the integrals I and II evaluated, we have a complete description of the structure of the Mach layer from high subsonic through supersonic velocities. Inserting these forms, the uniform solution for density is,

$$(8.13)\quad \tilde{\rho} = - \frac{(2\tilde{u}_o^2+5)\tau\, e^{\frac{1}{27}\frac{t^3 u_o \sin\theta}{h^2 L} - \frac{4}{9}\frac{t^2 u_o^2 \cos\theta}{L^2 \sin\theta} + \frac{8}{27}\frac{h\cos^3\theta}{L\sin^2\theta}}}{h^{1/3}\, 5\sin^{1/3} L^{2/3}\, 3^{1/3}\, 4^{1/3}} \times$$

$$A_i\left(\frac{1}{3^{4/3}}\left[\frac{t^2 u_o^{2/3}\sin^2\theta}{2^{2/3}h^{4/3}L^{2/3}} - \frac{4^{1/6} t u_o^{2/3}\cos\theta}{L^{2/3}h^{1/3}\sin^{1/3}\theta} + \frac{h^{2/3}u_o^{2/3}\, 4^{2/3}\cos^2\theta}{L^{2/3}\sin^{4/3}\theta}\right]\right)$$

where A_i is the Airy function described in Appendix A. Corresponding forms can easily be written for the remainder of the flow field. Because of the complicated forms we only exhibit two flow regimes which are expecially simple to represent. These are almost sonic flow, when $|\cos\theta| \leq O(L)$, and supersonic flow, when both $\cos\theta$ and $\sin\theta$ are bounded away from zero. Note though, that any case can be computed since the Airy integrals are tabulated, [11].

Supersonic Flow

In Appendix A the asymptotic forms of the integrals I and II are given in the supersonic limit, (A.25) and (A.26) respectively. These correspond to the asymptotic development of our Airy integral for large values of the argument, and in our present notation these are given by

$$(8.14) \qquad I \sim \frac{i \sin\theta \, L^{1/2} \, e^{\frac{-t^2 \tilde{u}_o \cos\theta}{hL}}}{4h^{1/2} \, u_o^{1/2} \, \cos^{1/2}\theta \, \pi^{1/2}}$$

and

$$(8.15) \qquad II \sim 0.$$

The latter is not given to lowest order since it occurs in combination with the integral I, in the flow field representation. Inserting these calculations into equations (8.4), (8.5), (8.9), (8.10), we have to lowest order

$$(8.16) \quad \left\{ \begin{aligned} \tilde{\rho} &= -\frac{\tau(2\tilde{u}_o^2 + 5)}{10\,\tilde{u}_o \cos\theta} \left(\frac{\tilde{u}_o \cos\theta}{4Lh\pi}\right)^{1/2} e^{\frac{-t^2 \tilde{u}_o \cos\theta}{4Lh}} \\ \tilde{T} &= \frac{2}{3}\,\tilde{\rho} \\ \tilde{u}_1 &= \frac{(2u_o^2 + 5)\tau}{6\tilde{u}_o^2 \cos\theta} \left(\frac{\tilde{u}_o \cos\theta}{4\pi hL}\right)^{1/2} e^{\frac{-t^2 \tilde{u}_o \cos\theta}{4Lh}} \\ \tilde{u}_2 &= \frac{-(2u_o^2 + 5)\tau}{6\tilde{u}_o^2 \sin\theta} \left(\frac{\tilde{u}_o \cos\theta}{4\pi hL}\right)^{1/2} e^{\frac{-t^2 u_o \cos\theta}{4Lh}} \end{aligned} \right.$$

A form which is in asymptotic agreement with the system (8.16) already exists in the literature, see [18], [19]. Examining the flow given by the equations given in (8.13) we see that the remarks made in Section 6 are justified. The thickness of the layer varies as $(hL)^{1/2}$, and the similarity variable is $(t^2 u_o \cos\theta/4Lh)$. When viewed as the asymptotic solution to the problem of gas flow past an object, the assumed linearization is valid when $\tau/(hL)^{1/2}$ is small compared to unity. On viewing the set (8.13) as a fundamental solution and passing to the limit of small L we see that the inviscid solutions of Section 4, (4.17-4.19) are recovered. It is interesting to note that the inviscid solutions are also obtained in passing to the limit of small h. This limit should be thought of as being carried out for a fixed but very small mean free path L. Physically, this process tells us that on approaching the boundary layer of an object, we approach the conditions of inviscid flow. Again, as in the wake of Section 6, the convergence to the Dirac δ-function is not uniform. For any small $L > 0$, we can go far enough away from the origin such that the flow is viscous. As a criterion for viscous flow we can choose that radial distance at which the Mach layer is double the body size. A simple calculation shows that this is $O(\mathcal{L}^2/L)$. This shows that even though the flow is inviscid outside the boundary layer, at sufficiently large distances the flow field again must be described by a viscous theory.

Sonic Flow: $|\cos\theta| \leq O(L)$

This case, as mentioned, also allows a simple representation. The appendix contains the representation of integrals I and II in the sonic limit. In this limit we can return to the x-y coordinate system. From (A.20) and (A.21) we get

$$(8.17) \qquad I = \frac{iL^{1/3}\, e^{-\frac{1}{27}\frac{x^3 y_o}{y^2 L}}}{3^{1/3}\, 4^{1/3} |y|^{1/3}\, \tilde{u}_o^{1/3}}\, A_i\left(\frac{x^2 \tilde{u}_o^{2/3}}{3^{4/3}\, 2^{2/3}\, y^{4/3}\, L^{2/3}}\right)$$

and

$$(8.18) \qquad II \sim \frac{1}{3}\frac{x}{y}\, I.$$

Equation (8.18) is approximate since a higher order term is neglected. The latter term is no longer negligible for large values of the argument of the Airy integral (outside the Mach layer). If a study of the full flow field is asked for, the remainder term must be included. Inserting the above forms into (8.4,8.5,8.9), we get the sonic flow field,

$$(8.19) \qquad \begin{pmatrix} \tilde{\rho} \\ \tilde{T} \\ \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -\frac{2}{3} \\ u_o \\ -\frac{u_o x}{3y} \end{pmatrix} \frac{5\tau\, e^{-\frac{1}{27}\frac{x^3 u_o}{y^2 L}}}{3^{4/3} 4^{1/3} |y|^{1/3} L^{2/3} \tilde{u}_o^{1/3}}\, A_i\left(\frac{x^2 u_o^{2/3}}{3^{4/3} 2^{2/3} y^{4/3} L^{2/3}}\right)$$

The expansion of the Airy function in the layer is given in the appendix, (A.17). In the above, use has been made of the relation, $\tilde{u}_o^{\,2} = 5/3$ which holds for sonic flow. Notice that $\tilde{u}_2$ is of a higher order; we keep it in order to exhibit it in its lowest order.

Again the conclusions drawn in Section 6 are verified. The thickness of the layer varies as $(y^2L)^{1/3}$ and the similarity variable is (x^3u_o/y^2L). For flow past an object, $\tau/(hL^2)^{1/3}$ must be small compared with unity in order to justify the assumed linearization. One also notices that, irrespective of the problem, the theory indicates that in order for the linearization to be correct, one must be further away from the body in transonic flow than in supersonic flow.

In order to examine the sonic flow field in the limit of vanishing mean free path, it is convenient for formal reasons to return to (8.4, 5, 9, 10). From these relations we see that the sonic flow field may be represented by

$$\begin{pmatrix} \tilde{\rho} \\ \tilde{T} \\ \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix} = \frac{5}{6} \begin{pmatrix} -\frac{1}{u_o}\frac{\partial}{\partial x} \\ -\frac{2}{3u_o}\frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} \phi \tag{8.20}$$

where

$$(8.21) \qquad \emptyset = \frac{2}{(2\pi)^2} \int \frac{e^{-imx + iny}\, dm\, dn}{(n^2 - L^2 \frac{im^3}{\tilde{u}_o})}$$

acts as the velocity potential. Although this is a divergent integral it has meaning in the sense of distributions, and the evaluation follows that of Appendix A. This yields

$$\emptyset = |y| \left\{ \frac{x\tilde{u}_o^{2/3}\, 2^{1/3}\, e^{-\frac{x^3\tilde{u}_o}{27y^2L}}}{3^{1/3}\, y^{4/3}\, L^{2/3}} A_i\left(\frac{x^2\tilde{u}_o^{2/3}}{3^{4/3}\, 2^{2/3}\, y^{4/3}\, L^{2/3}}\right) + \right.$$

$$\left. + \frac{3^{1/3} 4^{1/3} \tilde{u}_o^{1/3}}{y^{2/3}\, L^{1/3}}\, e^{-\frac{x^2\tilde{u}_o}{27y^2L}}\, A_i'\left(\frac{x^2\tilde{u}_o^{2/3}}{3^{4/3}\, 2^{2/3}\, y^{4/3}\, L^{2/3}}\right) \right\}$$

where the prime denotes differentiation with respect to the argument. In order to show that the solution (8.20) becomes the inviscid sonic solution of Section 4, we need only show that the bracket of equation (8.22) becomes a delta function in the limit of zero mean free path. We do this in the usual way, by letting the bracket operatr on some testing function $\psi(x)$. Consider the integral

$$(8.23) \qquad \int \frac{\emptyset(x)}{|y|} \psi(x) dx$$

taken across the origin between some definite limits. Under the transformation

$$(8.24) \qquad s = \frac{x\tilde{u}_o^{1/3}}{3^{2/3}\, 2^{1/3}\, L^{1/3}\, y^{2/3}}$$

this becomes

$$(8.25) \qquad 6\int s e^{-\frac{2s^3}{3}} A_i(s^2)\psi\left(\frac{L^{1/3}\, 3^{2/3}\, 2^{1/3}\, y^{2/3}}{\tilde{u}_o^{1/3}} s\right)ds +$$

$$+ 6\int e^{-\frac{2}{3}s^3} A_i'(s^2)\psi\left(\frac{L^{1/3}\, 3^{2/3}\, 2^{1/3}\, y^{2/3}}{\tilde{u}_o^{1/3}} s\right)ds .$$

Expanding the testing function in a Taylor series and passing to the limit, we have our result. Notice that again the convergence to the delta function is not uniform. The lowest order term after the delta function is $O(L^{1/3}\, y^{2/3})$. As in the case of supersonic flow and the wake before that, for any small but fixed L, we can go far enough away so that a viscous theory must be used. Then quadratic terms are small compared to linear terms and the embarrassment of infinite drag is removed.

9. Contribution of Lift and Heat Addition.

In Section 4 we restricted the particle source so that to lowest order it would act only as a drag singularity. This excluded the possibility of lift and heat addition. In general however, these two have to be included, buth for the asymptotic flow far from a body and in the fundamental solution. This section deals with the effect of the additional terms.

Returning to the source integrals, in place of equation (4.4) we define

$$(9.1) \qquad \int \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \tilde{\sigma}\, d\tilde{\xi} = u_o \begin{pmatrix} \tau \\ \lambda \end{pmatrix}$$

and

$$(9.2) \qquad \int \tilde{\xi}^2 \sigma\, d\tilde{\xi} = 2q\tilde{u}_o^{\,2} \ .$$

Previously the heat addition and lift terms, q and λ, where taken as zero. Inserting (9.1) and (9.2) into equation (4.1) our source terms become

$$(9.3) \qquad \sigma_2 \sim \frac{2\tilde{u}_o^2}{3}\,(q-\tau)$$

$$(9.4) \qquad \begin{pmatrix} \sigma_3^1 \\ \sigma_3^2 \end{pmatrix} \sim \tilde{u}_o \begin{pmatrix} \tau \\ \lambda \end{pmatrix}$$

We will consider the effects of lift and heat addition in the wake layer and Mach layer separately.

Wake:

The source term in the contracted system (7.3) now has the form

$$(9.5) \qquad \begin{pmatrix} 0 \\ \\ \frac{2\tilde{u}_o^2}{3}\,(q-\tau) \\ \\ k_1\tau\tilde{u}_o + k_2\lambda\tilde{u}_o \end{pmatrix}$$

From inspection of equation (7.3) we see that to the lowest order, again only the entropy source enters. This does not mean that the asymmetry of the lift is ineffectual in the wake. As we will see shortly the lift source produces an asymmetry in the wake velocity field. To this order then the density field is

$$(9.6) \qquad \tilde{\rho} = \frac{2}{5}\,\tilde{u}_o(\tau-q)\,\left(\frac{\tilde{u}_o}{4\pi Lx}\right)^{1/2} e^{-y^2u_o/4Lx}$$

and

$$(9.7) \qquad \tilde{T} = -\tilde{\rho}.$$

A simple calculation shows that the veolcity field is given by

$$(9.8) \qquad \begin{cases} \tilde{u}_1 = \tau H(x)\,\left(\frac{\tilde{u}_o}{4\pi Lx}\right)^{1/2} e^{-y^2\tilde{u}_o/4Lx} \\ \\ \tilde{u}_2 = \lambda H(x)\,\left(\frac{\tilde{u}_o}{4\pi Lx}\right)^{1/2} e^{-y^2\tilde{u}_o/4Lx} - \\ \qquad - \frac{4}{5}\,(\tau-q)\tilde{u}_o^2\,\frac{yH(x)}{x}\,\left(\frac{\tilde{u}_o}{4\pi Lx}\right)^{1/2} e^{-y^2\tilde{u}_o/4Lx} \end{cases}.$$

Mach Layer:

Returning to equation (8.2) we have to replace the source term by

$$(9.9)\qquad \begin{pmatrix} 0 \\ -\frac{2}{3} u_o^2 (\tau - q) \\ -u_o m(\tau \sin\theta - \lambda \cos\theta) \end{pmatrix}$$

A simple calculation shows that with (9.9), we need only replace $(2u_o^2 + 5)\tau$ by

$$(9.10)\qquad \left\{(2u_o^2 + 5)\tau - 2u_o^2 q - 5\lambda \cot\theta\right\}$$

in all the formulae to get the additional effect of lift and heat addition.

Appendix A: Evaluation of Integrals.

In Section 8, there occurred the two related integrals,

$$(A.1) \qquad I = \frac{1}{(2\pi)^2 u_o} \int \frac{e^{imt + inh} \, m \, dm \, dn}{\left(\frac{n^2}{L} + \frac{2mn \cot \theta}{L} - \frac{2im^3 \csc \theta}{u_o}\right)}$$

$$(A.2) \qquad II = \frac{1}{(2\pi)^2 u_o} \int \frac{e^{imt + inh} \, n \, dm \, dn}{\left(\frac{n^2}{L} + \frac{2mn \cot \theta}{L} - \frac{2im^3 \csc \theta}{u_o}\right)}$$

Under the transformation,

$$(A.3) \qquad \begin{cases} n = \frac{4u_o \nu}{L} , & h = \frac{L}{4u_o} h' \\ m = \frac{2u_o \mu}{L} , & t = \frac{L}{2u_o} t' \end{cases}$$

(A.1) and (A.2) become, respectively

$$(A.4) \qquad I = \frac{1}{(2\pi)^2} \int \frac{e^{i\mu t' + i\nu h'} \, \mu \, d\mu \, d\nu}{\nu^2 + \mu\nu \cot \theta - i\mu^3 \csc \theta}$$

$$(A.5) \qquad II = \frac{1}{2\pi^2} \int \frac{e^{i\mu t' + i\nu h'} \, \nu \, d\mu \, d\nu}{\nu^2 + \mu\nu \cot \theta - i\mu^3 \csc \theta}$$

Integral I.

Since $h' > 0$, we can complete the contour in the upper half complex ν-plane. We are therefore interested in those roots of the denominator which have a positive imaginary part. The roots of the denominator are given by

$$(\text{A.6}) \qquad \nu_{\pm} = - \frac{\mu[\cos\theta \pm \sqrt{4i\mu\sin\theta + \cos^2\theta}\,]}{2\sin\theta}$$

Choosing a cut just below the negative real axis of the complex μ-plane, we see the ν_+ is always in the upper half plane in the first Riemann sheet. The integration in the ν-plane gives

$$(\text{A.7}) \qquad I = \frac{i\sin\theta}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\mu t' + i\nu_+ h'}\, d\mu}{\sqrt{\cos^2\theta + 4i\mu\sin\theta}}$$

Next taking

$$(\text{A.8}) \qquad \sqrt{\cos^2\theta + 4\mu i\sin\theta} = \frac{2\sin^{2/3}\theta}{h'^{1/3}}\,\zeta$$

we get

$$(\text{A.9}) \qquad I = \frac{\sin^{2/3}\theta\; e^{-t'\cos^2\theta/4\sin\theta + h'\cos^3\theta/8\sin^2\theta}}{2\pi h'^{1/3}} \times$$

$$\int_C e^{\zeta^3 + \zeta^2\left(\frac{t'\sin\theta}{h'^{2/3}} - \frac{h'^{1/3}\cos\theta}{2\sin^{2/3}\theta}\right) - \zeta\frac{h'^{2/3}\cos^2\theta}{4\sin^{4/3}\theta}}\, d\zeta$$

with the path C shown in Figure A.1

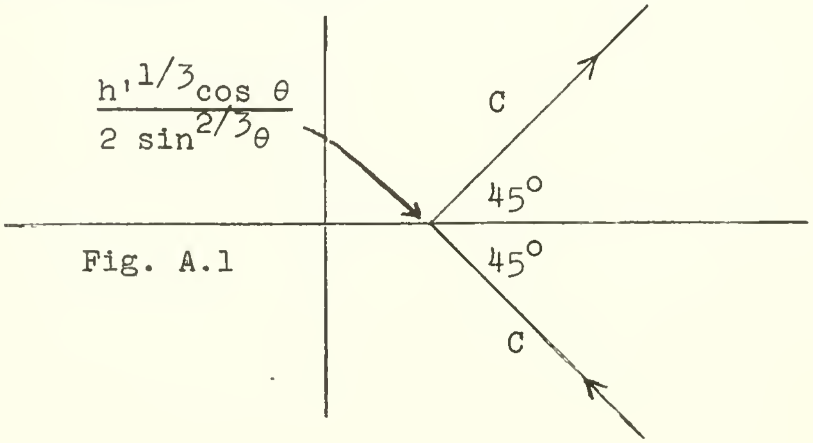

Fig. A.1

Finally we perform the translation,

$$(A.10) \qquad \zeta = \frac{s}{3^{1/3}} - \frac{1}{3}\left(\frac{t'\sin\theta}{h'^{2/3}} - \frac{h'^{1/3}\cos\theta}{2\sin^{2/3}\theta}\right)$$

to get

$$(A.11) \qquad I = \frac{i\sin^{2/3}\theta}{h'^{1/3}3^{1/3}}\, e^{\frac{2}{27}\frac{t'^3\sin\theta}{h'^2} - \frac{1}{9}\frac{t'^2\cos^2\theta}{\sin\theta} + \frac{2}{27}\frac{h'\cos^3\theta}{\sin^2\theta}}\, A_i(x).$$

$A_i(x)$ is an Airy integral, given by

$$(A.12) \qquad A_i(x) = \frac{1}{2\pi i}\int_C e^{\sigma^3/3 - x\sigma}\, d\sigma,$$

see [11]. The path C is of the type shown in Figure A.1. For the case at hand

$$(A.13) \qquad x = \frac{1}{3^{4/3}}\left(\frac{t'^2\sin\theta}{h'^{4/3}} - \frac{t'\cos\theta}{h'^{1/3}\sin^{1/3}\theta} + \frac{h'^{2/3}\cos^2\theta}{\sin^{4/3}\theta}\right).$$

Integral II

This integration follows the methods used in the integration of I. Carrying out the integration in the complex ν-plane we get

$$(A.14) \qquad II = \frac{i}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\mu t' + i\nu_+ h'}(-\cos\theta + \sqrt{\cos^2\theta + 4i \quad \sin\theta})\, d\mu}{\sqrt{\cos^2\theta + 4i\mu\sin\theta}}$$

$$= -\cot\theta\, I + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\mu t' + i\nu_+ h'}\, d\mu$$

Considering only the second part,

$$(A.15) \qquad II' = \frac{i}{2\pi} \int_{-\infty}^{\infty} e^{i\mu t' + i\nu_+ h'} \, d\mu$$

and introducing the changes in variables given by (A.8) and (A.10), we get

$$(A.16) \qquad II' = -\frac{2}{3} \frac{1}{\sin^{1/3}\theta \; h'^{1/3}} \left(\frac{t' \sin\theta}{h'^{2/3}} - \frac{h'^{1/3}}{2} \frac{\cos\theta}{\sin^{2/3}\theta}\right) I -$$

$$- \frac{2i \sin^{1/3}\theta}{h'^{2/3} 3^{2/3}} \, e^{\frac{2}{27} \frac{t'^3}{h'^2} \sin\theta - \frac{1}{9} \frac{t'^2}{h'} \cos\theta - \frac{1}{9} \frac{t'^2 \cos^2\theta}{\sin\theta} + \frac{2}{27} \frac{h' \cos^3\theta}{\sin^2\theta}} \; A_i'(x).$$

Here, x is again given by (A.13), and $A_i'(x)$ denotes the derivative of the Airy integral, (A.12).

Expansions of the Airy integral may be found in [11]. The power series expansion is given by

$$(A.17) \qquad A_i(x) = \frac{1}{3^{2/3}(-\frac{1}{3})!} \left(1 + \frac{1}{3!} x^3 + \ldots\right)$$

$$- \frac{1}{3^{1/3}(-\frac{2}{3})!} \left(x + \frac{2}{4!} x^4 + \ldots\right)$$

and the asymptotic expansion is given by

$$(A.18) \qquad A_i(x) \sim \frac{e^{-\xi}}{2\pi^{1/2} x^{1/4}} \left(1 - \frac{3.5}{216} \frac{1}{\xi} + \ldots\right)$$

with

$$\xi = \frac{2}{3}x^{3/2}. \tag{A.19}$$

In connection with almost sonic flow, the integrals may be written to lowest order as

$$I = \frac{i}{h'^{1/3}\,3^{1/3}}\, e^{\frac{2}{27}\frac{t'^3}{h'^2}}\, A_i(x) \tag{A.20}$$

and

$$II \sim -\frac{2}{3}\frac{t'}{h'}\, I, \tag{A.21}$$

with

$$x = \frac{1}{3^{4/3}}\frac{t'^2}{h'^{4/3}}\,. \tag{A.22}$$

Since the argument of A_i is $O(1)$ the power series expansion, (A.17), may be used in this case. In connection with the supersonic limit, when both $\cos\theta$ and $\sin\theta$ are bounded away from zero, we consider the integrals I and II as $L \longrightarrow 0$. Then with

$$x \quad \frac{h'^{2/3}\cos^2\theta}{3^{4/3}\sin^{4/3}\theta} \tag{A.25}$$

and

$$\xi = \frac{2}{27}\frac{h'\cos^3\theta}{\sin^2\theta} - \frac{1}{9}\frac{t'\cos^2\theta}{\sin\theta} + \frac{5}{36}\frac{t'^2\cos\theta}{h'} - \\ -\frac{1}{18}\frac{t'^3\sin\theta}{h'^2}\,, \tag{A.24}$$

we get, using (A.18), that

$$I \sim \frac{i \sin \theta}{2h'^{1/2} \cos \theta^{1/2} \pi^{1/2}} e^{-\frac{1}{4} \frac{t'^2 \cos \theta}{h'}} \tag{A.25}$$

and

$$II \sim 0. \tag{A.26}$$

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